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ON THE STRUCTURE OF PERIODIC ARITHMETICAL MAPS

Zhi-Wei Sun

Department of Mathematics
Nanjing University
Nanjing 210093
The People's Republic of China
E-mail: zwsun@nju.edu.cn
Homepage: http://pweb.nju.edu.cn/zwsun

Abstract. This is an introduction to the algebraic theory of periodic arithmetical maps. The topic is connected with number theory, combinatorics, algebra and analysis.

1. Various Backgrounds

I. Combinatorial Background. For $n \in \mathbb{Z}^+ = \{1, 2, 3, \cdots\}$ and $a \in R(n) = \{0, 1, \cdots, n-1\}$, let

$$a(n) = a + n\mathbb{Z} = \{x \in \mathbb{Z} : \ x \equiv a \pmod n\} = \{\cdots, a-n, a, a+n, \cdots\}$$

and call it a residue class with *modulus* n. For a finite system

$$A = \{a_s(n_s)\}_{s=1}^k \tag{1.1}$$

of such residue classes, we define its *covering function* $w_A : \mathbb{Z} \to \mathbb{N} = \{0, 1, 2, \cdots\}$ by

$$w_A(x) = |\{1 \le s \le k : x \equiv a_s \pmod{n_s}\}|. \tag{1.2}$$

If $w_A(x) \ge m$ for all $x \in \mathbb{Z}$, then we call A an *m-cover* of $\mathbb{Z}$; if $w_A(x) = m$ for all $x \in \mathbb{Z}$, then we call A an *exact m-cover* of $\mathbb{Z}$. An exact 1-cover is also called a disjoint cover, it is just a partition of $\mathbb{Z}$ into finitely many residue classes. Clearly $\{r(n)\}_{r=0}^{n-1}$ is a disjoint cover for any $n \in \mathbb{Z}^+$, also the system

$$\{0(2),\ 0(3),\ 1(4),\ 5(6),\ 7(12)\}$$

form a cover of $\mathbb{Z}$ with distinct moduli. The combinatorial concepts of cover and disjoint cover were first introduced by P. Erdős in the 1930's, he often considered them as his most favorite invention. In general we can assign the sth residue class $a_s(n_s)$ a number weight λ_s. Thus we consider the system

$$\mathcal{A} = \{\langle \lambda_s, a_s, n_s\rangle\}_{s=1}^{k} \tag{1.3}$$

of triples and define its covering function $w_{\mathcal{A}} : \mathbb{Z} \to \mathbb{C}$ as follows:

$$w_{\mathcal{A}}(x) = \sum_{\substack{s=1 \\ x \in a_s(n_s)}}^{k} \lambda_s. \tag{1.4}$$

II. Number-theoretic Background. An arithmetical function $\psi : \mathbb{Z} \to \mathbb{C}$ is said to be periodic modulo $n \in \mathbb{Z}^+$ if $\psi(x+n) = \psi(x)$ for all $x \in \mathbb{Z}$. If $\psi_1, \cdots, \psi_k$ are arithmetical functions having the smallest positive periods $n_1, \cdots, n_k$ respectively, what is the smallest positive period of the function $\psi = \psi_1 + \cdots + \psi_k$? The study of periodic arithmetical functions seems to be a task of number-theorists.

III. Algebraic Background. The set $P(\mathbb{C})$ of all periodic arithmetical functions $\psi : \mathbb{Z} \to \mathbb{C}$, forms a linear space over the complex field $\mathbb{C}$. What are those linear transformations T of $P(\mathbb{C})$? For such a T clearly $T(\psi)$ should depend on the smallest positive period $n(\psi)$ of $\psi \in P(\mathbb{C})$. However, for general $\psi_1, \psi_2 \in P(\mathbb{C})$ we don't know $n(\psi_1 + \psi_2)$ exactly even if $n(\psi_1)$ and $n(\psi_2)$ are known. This is the main difficulty, which makes the problem more interesting and challenging!

IV. Analytic Background. In analysis there are many important special functions. The Bernoulli polynomials $B_n(x)$ $(n = 0, 1, 2, \cdots)$ are defined by

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!}.$$

Raabe's theorem asserts that if $m \in \mathbb{N}$ and $n \in \mathbb{Z}^+$ then

$$\sum_{r=0}^{n-1} B_n\left(x+\frac{r}{n}\right) = n^{1-m}B_m(nx).$$

The well-known Γ-function is given by

$$\Gamma(x) = \lim_{n\to+\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)},$$

a multiplication formula of Gauss states that

$$\prod_{r=0}^{n-1}\Gamma\left(z+\frac{r}{n}\right) = (2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nz}\Gamma(nz) \quad \text{for } n\in\mathbb{Z}^+ \text{ and } z\neq 0,-1,-2,\cdots.$$

The Hurwitz zeta function is defined as follows:

$$\zeta(s,z) = \sum_{n=0}^{\infty}\frac{1}{(n+z)^s} \quad (\mathrm{Re}(s)>1).$$

Observe that $\zeta(s,1)$ is just the Riemann zeta function $\zeta(s)$.

2. A Unified Theory

Let M be an additive abelian group. For

$$\mathcal{A} = \{\langle\lambda_s, a_s, n_s\rangle\}_{s=1}^{k} \quad \text{with } \lambda_s\in M, \tag{2.1}$$

the *covering map*

$$w_{\mathcal{A}}(x) = \sum_{\substack{s=1\\ x\in a_s(n_s)}}^{k} \lambda_s \quad (x\in\mathbb{Z}), \tag{2.2}$$

is periodic modulo the least common multiple $N=[n_1,\cdots,n_k]$ of all the moduli $n_1,\cdots,n_k$. The set $P(M)$ of all periodic maps $\psi:\mathbb{Z}\to M$ just consists of such covering maps, because any $\psi\in P(M)$ is the covering map of the system $\{\langle\psi(r),r,n\rangle\}_{r=0}^{n-1}$. Thus the concepts of cover and covering function help us to understand periodic arithmetical functions. Two systems $\mathcal{A}_1,\mathcal{A}_2$ in the form (2.1) are said to be equivalent (we denote this by $\mathcal{A}_1 \sim \mathcal{A}_2$) if they have the same covering map.

Let $\Omega = \bigcup_{n\in\mathbb{Z}^+} \mathbb{Z}/n\mathbb{Z}$, and $F(M)$ denote the set of all maps from Ω to M. A map $f \in F(M)$ is said to be *equivalent* if

$$\sum_{j=0}^{n-1} f(a+jd+nd\mathbb{Z}) = f(a+d\mathbb{Z}) \ \text{ for all } a\in\mathbb{Z} \text{ and } d,n\in\mathbb{Z}^+. \tag{2.3}$$

Note that $\{a+jd(nd)\}_{j=0}^{n-1} \sim \{a(d)\}$. Those equivalent $f \in F(M)$ forms a subgroup $E(M)$ of the abelian group $F(M)$ under the functional addition.

Let R be a ring. Then $F(R)$ forms a ring with subring $E(R)$ under the addition and the convolution $*$ defined below:

$$f*g(a+n\mathbb{Z}) = \sum_{r=0}^{n-1} f(r+n\mathbb{Z})g(a-r+n\mathbb{Z}) \ \ (a\in\mathbb{Z},\ n\in\mathbb{Z}^+). \tag{2.4}$$

If R has the identity 1, then $F(R)$ and $E(R)$ have the identity

$$e(a+n\mathbb{Z}) = \begin{cases} 1 & \text{if } n\mid a, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

In 1989 Sun [Nanjing Univ. J. Math. Biquarterly] introduced the concept of uniform function.

Definition. Let M be an additive abelian group, and F be a map from a subset of $\mathbb{C}\times\mathbb{C}$ into M. If for any ordered pair $\langle x,y\rangle$ in the domain $\mathrm{Dom}(F)$ of F and each positive integer n, we have

$$\left\{\left\langle \frac{x+r}{n}, ny\right\rangle : r=0,1,\cdots,n-1\right\} \subseteq \mathrm{Dom}(F) \tag{2.6}$$

and

$$\sum_{r=0}^{n-1} F\left(\frac{x+r}{n}, ny\right) = F(x,y), \tag{2.7}$$

then we call F a *uniform map* (to M).

If F is a uniform map to M, then for any $\langle x, y\rangle \in \mathrm{Dom}(F)$, the function $f(a+n\mathbb{Z}) = F(\frac{x+a}{n}, ny)$ ($a, n \in \mathbb{Z}$ and $0 \le a < n$) is an equivalent map to M. Conversely, if $f \in E(M)$ then the function $F(x, y) = f(xy + y\mathbb{Z})$ ($y \in \mathbb{Z}^+$ and $xy \in \mathbb{Z}$) is a uniform map to M.

If $F(x, y)$ is a uniform function, then so is $F^-(x, y) = F(\{x\}, y)$. Uniform functions are rich in examples.

An identity of Hermite is as follows:

$$\sum_{r=0}^{n-1}\left[x+\frac{r}{n}\right] = [nx] \qquad \text{for } x \in \mathbb{R} \text{ and } n \in \mathbb{Z}^+.$$

This shows that $[\ ](x, y) = [x]$ is a uniform function.

Raabe's theorem just says that the function $b_m(x, y) = B_m(x)y^{m-1}$ is a uniform map to C. Note that $b_0(x, y) = 1/y,\ b_1(x, y) = x-1/2$ and $b_1(x, y)-b_1^-(x, y) = [x]$.

For $s \in \mathbb{C}$ with $Re(s) > 1$, we can easily verify that $\zeta_s(x, y) = y^{-s}\zeta(s, x)$ is a uniform function where $x, y > 0$.

Sun observed in 1989 that the multiplication formula of Gauss is actually equivalent to the following statement: $\log\gamma(x, y)$ is a uniform function where

$$\gamma(x, y) = \Gamma(x)y^{x-\frac{1}{2}}/\sqrt{2\pi} \qquad \text{for } x \neq 0, -1, -2, \cdots \text{ and } y > 0. \tag{2.8}$$

$\log(2\sin\pi x)$ and $\frac{1}{y}\cot\pi x$ are also uniform functions where $x \notin \mathbb{Z}$ and $y \neq 0$.

In view of various different kinds of uniform functions, it seems that we can not give a unified form for equivalent functions. However, we have

Theorem 2.1 [Z. W. Sun , Adv. Math. China, 1989; J. Algebra, 2001]**.** *For a function $f : \Omega \to \mathbb{C}$, $f \in E(\mathbb{C})$ if and only if f has the following form:*

$$f(a+n\mathbb{Z}) = \frac{1}{n}\sum_{m=0}^{n-1}\psi\left(\frac{m}{n}\right)e^{2\pi i\frac{m}{n}a} \tag{2.9}$$

where ψ is a function from $\mathbb{Q}\cap[0,1)$ to $\mathbb{C}$.

By Theorem 2.1 and the well-known Ramanujan sum, we have the following example of equivalent functions:

$$\Phi(a+n\mathbb{Z})=\begin{cases}1/\varphi(n) & \text{if } (a,n)=1,\\ 0 & \text{otherwise.}\end{cases}$$

Let R be a commutative ring with identity and M, N be R-modules. The set of all R-module homomorphisms from M to N forms an R-module in a natural way and we denote it by $\mathrm{Hom}_R(M,N)$. Those $T\in \mathrm{Hom}_R(P(M),P(N))$ commute with the shift operator E (i.e. $T(E\psi)=E(T(\psi))$ where $E\psi(x)=\psi(x+1)$) form a submodule $\mathrm{Hom}'_R(P(M),P(N))$ of $\mathrm{Hom}_R(P(M),P(N))$.

Why shall we study equivalent maps? Their importance can be seen from the following theorem.

Theorem 2.2 [Z. W. Sun, J. Algebra, 2001]**.** *Let R be a ring with identity, and M be a left R-module.*

(i) *If R is commutative, then the R-module $\mathrm{Hom}'_R(P(R),P(M))$ is isomorphic to the R-module $E(M)$.*

(ii) *$P(M)$ forms an $E(R)$-module with respect to the natural addition and the scalar multiplication $\circ$ defined below:*

$$f\circ\psi(x)=\sum_{r=0}^{n-1}f(r+n\mathbb{Z})\psi(x-r)\quad \textit{where } n\in\mathbb{Z}^+ \textit{ is a period of } \psi. \tag{2.10}$$

As $f\in E(R)$ the right hand side does not depend on the choice of a period of ψ.

(iii) *If R is commutative, then $\mathrm{Hom}'_R(P(R),P(R))$ consists of those $T_f:\psi\mapsto f\circ\psi$ with $f\in E(R)$, and the ring $\mathrm{Hom}'_R(P(R),P(R))$ is isomorphic to $E(R)$.*

Corollary 2.1 [Z. W. Sun, Nanjing Univ. J. Math. Biquarterly, 1989]**.** *Let F be a map to C with $\mathrm{Dom}(F)\subseteq\mathbb{C}\times\mathbb{C}$ such that (2.6) holds for any $\langle x,y\rangle\in\mathrm{Dom}(F)$ and $n\in\mathbb{Z}^+$. Then the following two statements are equivalent:*

(a) *F is a uniform map to $\mathbb{C}$.*

(b) *Whenever*

$$\{\langle\lambda_s, a_s, n_s\rangle\}_{s=1}^k \sim \{\langle\mu_t, b_t, m_t\rangle\}_{t=1}^l \tag{2.11}$$

(*with* $\lambda_s, \mu_t \in \mathbb{C}$, $0 \leqslant a_s < n_s$, $0 \leqslant b_t < m_t$), *we have*

$$\sum_{s=1}^k \lambda_s F\left(\frac{x+a_s}{n_s}, n_s y\right) = \sum_{t=1}^l \mu_t F\left(\frac{x+b_t}{m_t}, m_t y\right) \quad \textit{for } \langle x,y\rangle \in \mathrm{Dom}(F). \tag{2.12}$$

Proof. (b) implies (a) because $\{r(n)\}_{r=0}^{n-1} \sim \{0(1)\}$.

(a)⇒(b). Suppose that (2.11) holds. Then $\sum_{s=1}^k \lambda_s\psi_s(z) = \sum_{t=1}^l \mu_t\chi_t(z)$ where $\psi_s(z) = 1$ or 0 according to whether $n_s \mid z+a_s$ or not, and $\chi_t(z) = 1$ or 0 according to whether $m_t \mid z+b_t$ or not. Fix $\langle x,y\rangle \in \mathrm{Dom}(F)$ and let $f(a+n\mathbb{Z}) = F(\frac{x+a}{n}, ny)$ $(0 \le a < n)$. Note that

$$T_f(\psi_s)(z) = f\circ\psi_s(z) = \sum_{r=0}^{n_s-1} f(r+n_s\mathbb{Z})\psi_s(z-r) = f(z+a_s+n_s\mathbb{Z}).$$

Similarly, $T_f(\chi_t)(z) = f(z+b_t+m_t\mathbb{Z})$. So

$$\sum_{s=1}^k \lambda_s f(a_s+n_s\mathbb{Z}) = \sum_{t=1}^l \mu_t f(b_t+m_t\mathbb{Z}).$$

The proof is now complete. □

Actually Corollary 2.1 can be stated in a more abstract form.

Corollary 2.2 [Z. W. Sun, Nanjing Univ. J. Math. Biquarterly, 1989]**.** *Whenever* $A = \{a_s(n_s)\}_{s=1}^k \sim B = \{b_t(m_t)\}_{t=1}^l$ $(0 \le a_s < n_s$ *and* $0 \le b_t < m_t)$, *we have*

$$\prod_{s=1}^k \Gamma\left(\frac{x+a_s}{n_s}\right) n_s^{(x+a_s)/n_s-1/2} = (2\pi)^{(k-l)/2}\prod_{t=1}^l \Gamma\left(\frac{x+b_t}{m_t}\right) m_t^{(x+b_t)/m_t-1/2}$$

for $x \ne 0, -1, -2, \cdots$.

Proof. This is because $\log(\Gamma(x)y^{x-1/2}/\sqrt{2\pi})$ is a uniform function. The result can be viewed as a generalization of Gauss' multiplication formula. □

3. On Sums of Periodic Arithmetical Functions

Applying the theory in Section 2, Sun was able to obtain the following result.

Theorem 3.1 [Z. W. Sun, J. Algebra, 240(2001)]. *Let $\lambda_1, \cdots, \lambda_k \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $\xi_1, \cdots, \xi_k$ be distinct roots of unity. Then the smallest (positive) period of the arithmetical function $\psi(x) = \sum_{s=1}^k \lambda_s \xi_s^x$, coincides with $[n_1, \cdots, n_k]$ where n_s is the least $n \in \mathbb{Z}^+$ with $\xi_s^n = 1$ (i.e., ξ_s is a primitive n_sth root of unity).*

For system $A = \{a_s(n_s)\}_{s=1}^k$, the arithmetic average of $w_A(x)$ in a period is

$$\frac{1}{N}\sum_{x=0}^{N-1} w_A(x) = \frac{1}{N}\sum_{s=1}^{k} |\{0 \le x < N : x \in a_s(n_s)\}| = \frac{1}{N}\sum_{s=1}^{k}\frac{N}{n_s} = \sum_{s=1}^{k}\frac{1}{n_s}$$

where $N = [n_1, \cdots, n_k]$. Thus, $\sum_{s=1}^k 1/n_s \ge m$ if A forms an m-cover; and $\sum_{s=1}^k 1/n_s = m$ if A is an exact m-cover.

Theorem 3.2. *Let $A = \{a_s(n_s)\}_{s=1}^k$ and $n_0 \in \mathbb{Z}^+$ be the smallest positive period of $w_A(x)$.*

(i) [Z. W. Sun, Acta Arith. 1995; Trans. Amer. Math. Soc. 1996] *$w_A(x)$ takes the least value $m(A) = \min_{x\in\mathbb{Z}} w_A(x)$ when x ranges over an interval $[a, a+|S(A)|)$ of length $|S(A)|$ where*

$$S(A) = \left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\} : I \subseteq \{1, \cdots, k\}\right\}. \tag{3.1}$$

(ii) [Z. W. Sun, Combinatorica, in press] *$M(A) = \max_{x\in\mathbb{Z}} w_A(x)$ can be expressed in the form $\sum_{s=1}^k m_s/n_s$ where $m_1, \cdots, m_k \in \mathbb{Z}^+$. We also have*

$$\left\{\sum_{s\in I}\frac{1}{n_s} : I \subseteq \{1, \cdots, k-1\}\right\} \supseteq \left\{\frac{r}{n_k} : r = 0, 1, \cdots, \frac{n_k}{(n_0, n_k)} - 1\right\}. \tag{3.2}$$

(iii) [Z. W. Sun, Acta Arith. 1995, 1997] *Suppose that $n_0 = 1$, i.e. A forms an exact m-cover for some $m \in \mathbb{Z}^+$. Then, for any $\emptyset \ne J \subset \{1, \cdots, k\}$, there exists an $I \subseteq \{1, \cdots, k\}$ with $I \ne J$ such that*

$$\sum_{s\in I}\frac{1}{n_s} = \sum_{s\in J}\frac{1}{n_s}. \tag{3.3}$$

For $a = 0, 1, 2, \cdots$ *and* $t = 1, \cdots, k$, *we have*

$$\left|\left\{I \subseteq \{1, \cdots, k\} : t \notin I \ \&\ \sum_{s \in I} \frac{1}{n_s} = \frac{a}{n_t}\right\}\right| \geq \binom{m-1}{[a/n_t]} \tag{3.4}$$

where the lower bounds are best possible. The number of solutions of the equation

$$\sum_{s=1}^{k} \frac{x_s}{n_s} = c \quad \text{with } 0 \leqslant x_s < n_s \text{ for } s = 1, \cdots, k, \tag{3.5}$$

is the sum of finitely many (not necessarily distinct) prime factors of $n_1, \cdots, n_k$ *if* $c \neq 0, 1, 2, \cdots$, *and at least* $\binom{k-m}{n}$ *if* c *equals a nonnegative integer* n.