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RECENT PROGRESS ON COVERS OF THE INTEGERS AND THEIR APPLICATIONS

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1. Examples of Covers and Selected Applications

In 1849 A. de Polignac asked whether each odd integer greater than one can be expressed in the form $2^n + p$ where n is a nonnegative integer and p is 1 or a (positive) prime [actually Euler had already noted the counterexample 959]. Using the Brun sieve, in 1934 N.P. Romanoff proved that a positive proportion of the odd integers may be written in this way. On the other hand, by means of cover of the ring $\mathbb{Z}$ of the integers, in 1930's P. Erdős constructed a residue class of odd numbers which contains no integers of the desired form.

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\mathbb{Z}^+ = \mathbb{N} \setminus \{0\}$. For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ we call

$$a(n) = a + n\mathbb{Z} = \{x \in \mathbb{Z} : \ x \equiv a \pmod{n}\}$$

a *residue class* with *modulus* n or an *arithmetic sequence* with *common difference* n. A finite system

$$A = \{a_s(n_s)\}_{s=1}^{k} \tag{1.1}$$

of such classes is said to be a *cover* (of $\mathbb{Z}$) if every integer belongs to some residue classes in A. Clearly the *covering function*

$$w_A(x) = |\{1 \leqslant s \leqslant k : x \in a_s(n_s)\}| \tag{1.2}$$

is periodic modulo

$$N = [n_1, \cdots, n_k]. \tag{1.3}$$

Let p be a prime and $n \in \mathbb{Z}^+$. We say that p is a *primitive prime divisor* of $2^n - 1$ if

$$p \mid 2^n - 1 \ \text{ but } p \nmid 2^m - 1 \text{ for } 1 \leqslant m < n. \tag{1.4}$$

It is known that $2^n - 1$ has a primitive prime divisor if $n \neq 1, 6$. [Observe that $2^6 - 1 = 3^2 \times 7 = (2^2 - 1)^2(2^3 - 1)$.]

Lemma 1.1. *Let* (1.1) *be a cover of* $\mathbb{Z}$ *and* $p_1, \cdots, p_k$ *distinct prime divisors of* $2^{n_1} - 1, \cdots, 2^{n_k} - 1$ *respectively. Then, for any* x *in the residue class* $T(A) = \cap_{s=1}^{k} 2^{a_s}(p_s)$, *both* $x - 2^n$ *and* $x2^n - 1$ *always have prime divisors in* $\{p_1, \cdots, p_k\}$ *where* n *runs over* $\mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. As (1) forms a cover of $\mathbb{Z}$, there exist $1 \leqslant i, j \leqslant k$ such that $n \equiv a_i \pmod{n_i}$ and that $-n \equiv a_j \pmod{n_j}$. Since $p_i \mid 2^{n_i} - 1$ and $p_j \mid 2^{n_j} - 1$, $2^n \equiv 2^{a_i} \equiv x \pmod{p_i}$ and $2^n x \equiv 2^{n+a_j} \equiv 1 \pmod{p_j}$. This ends the proof. $\square$

Theorem 1.1. (i) (P. Erdős, 1950) *Let*

$$A_1 = \{0(2),\ 0(3),\ 1(4),\ 3(8),\ 7(12),\ 23(24)\}. \tag{1.5}$$

Those $x \in T(A_1)$ *satisfying additional congruences*

$$x \equiv 1 \pmod 2 \quad \text{and} \quad x \equiv 3 \pmod{31}$$

cannot be written in the form $2^n + p$ *where* $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$ *and* p *is a prime.*

(ii) (J.L. Selfridge) $78557 \times 2^n + 1$ $(n \in \mathbb{N})$ *are always divisible by one of* $3, 5, 7, 13, 19, 37, 73$.

Proof. i) Select a primitive prime divisor p for each $2^n - 1$ with n being a modulus in A_1. Then we get the set $P = \{3, 7, 5, 17, 13, 241\}$ of 5 primitive prime divisors. Let $x \in T(A_1) \cap 1(2) \cap 3(31)$. In view of Lemma 1.1 it suffices to show that x cannot be expressed in the form $2^n + p$ where $n \in \mathbb{N}$ and $p \in P$. Note that $|\{2^n + p \bmod 31 : n \in \mathbb{N} \ \&\ p \in P\}|$ is not more than $5 \times 6 = 30$. So, there is an $r \in \mathbb{Z}$ such that $2^n + p \not\equiv r \pmod{31}$ for all $n \in \mathbb{N}$ and $p \in P$, it can be checked that we may take $r = 3$. Since $x \equiv 3 \pmod{31}$, x cannot be written in the form $2^n + p$ where $n \in \mathbb{N}$ and p is a prime.

ii) Simply apply Lemma 1.1 to cover

$$A_2 = \{0(2),\ 2(3),\ 3(4),\ 6(9),\ 1(12),\ 3(18),\ 9(36)\} \tag{1.6}$$

and note that $T(A_2) = -78557(70050435)$. $\square$

Recently I and my student Si-Man Yang [Proc. Edinburgh Math. Soc., 45(2002)] improved Erdős' result as follows: For any integer $c \in [-3150, 20054)$ divisible by none of 3,5,7,13,17,241, the residue class $T(A_1) = 20036812(\text{mod } 5592405)$ contains no integers of the form $2^n + cp$ where $n \geqslant 0$ is an integer and p is a prime.

Inspired by the work of Erdős and using covers

$$B_1 = \{1(2),\ 0(4),\ 6(8),\ 10(12),\ 10(16),\ 18(24),\ 2(48)\} \tag{1.7}$$

and

$$\begin{aligned} B_2 = \{&0(2),\ 0(3),\ 2(5),\ 5(9),\ 3(10),\ 4(15), \\ &11(18),\ 1(20),\ 25(30),\ 17(36),\ 35(36),\ 31(60)\} \end{aligned} \tag{1.8}$$

($2^{36} - 1$ has two primitive prime divisors 37 and 109), in 1975 F. Cohen and J.L. Selfridge [CS] observed that the 26-digit prime

$$M = 47867742232066880047611079 \tag{1.9}$$

plus or minus a power of 2 can never be a prime, but they gave no reasons why additional congruences (similar to $x \equiv 3 \pmod{31}$ in the proof of Erdős) can be avoided there. (See their Theorem 1 and its proof.) They went to deduce (in their Theorem 2) that there exist odd numbers not of the form $\pm 2^a \pm p^b$ where p is a prime, $a, b \in \mathbb{N}$ and any choice of signs may be made. To find the least positive odd integer having the property is an interesting open problem. (Cf. section A19 of R.K. Guy's book *Unsolved Problems in Number Theory.*) Cohen and Selfridge showed that the 94-digit number

6120669906067276778092115601756625481957616163 1

-9229817343685493345124067417420946855899932656 9

is indeed not of the form $\pm 2^a \pm p^b$.

By introducing a new method to avoid a bunch of extra congruences, we are able to prove

Theorem 1.2. (i) (Zhi-Wei Sun, preprint) *There are infinitely many odd integers x such that neither x nor $x + 448$ can be written in the form $2^n \pm p^\alpha$ where p^α is a prime power. Also, neither $78557 + 2^n$ nor $78557 \times 2^n + 1$ can be a prime power.*

(ii) (Zhi-Wei Sun, Proc. Amer. Math. Soc., 128(2000), no. 4) *Any integer x in the residue class*

$$1(2) \cap T(B_1) \cap T(B_2) = M(66483084961588510124010691590) \tag{1.10}$$

with the 29-digit modulus being the product of primes

2, 3, 5, 7, 11, 13, 17, 19, 31, 37, 41, 61, 73, 97, 109, 151, 241, 257, 331,

cannot be written in the form $\pm p^a \pm q^b$ where p, q are primes, $a, b \in \mathbb{N}$ and any choice of signs may be made.

In Guy's book there is a list of 35 positive integers x less than 78557 for which $x2^n+1$ are composite for all $n \leqslant 50000$. But some of them don't possess the strong property described in the latter part of Theorem 1.2(i), for example the second number is $5297 = 73^2 - 2^5$, and the number 13787 plus 2 is a prime. We conjecture that 78557 is the smallest positive integer x such that both $x + 2^n$ and $x2^n + 1$ always have at least two distinct odd prime divisors.

I and Yun-Zhi Zou (at Sichuan Univ.) have used computer to show that any positive integer less than 2^{25} is the sum or difference of two prime powers. Since M is prime to the 29-digit modulus in Theorem 1.2(ii), with the help of Dirichlet's theorem there are infinitely many primes p such that $p + 2^n$ and $|p - 2^n|$ are both composite for all $n = 0, 1, 2, \cdots$. This gives an affirmative answer to the question raised by M.V. Vassilev-Missana.

Y.-G. Chen constructed a cover with no modulus being a power of 2 (actually Crocker had such an example in 1971) and employed a result of A. Baker to show

Theorem 1.3 (Y.G. Chen, Proc. Amer. Math. Soc. 128(2000), no. 6)**.** *The set of positive odd integers which have no representation of the form $2^n \pm p^\alpha q^\beta$ (where p, q are distinct odd primes and $n, \alpha, \beta \in \mathbb{N}$), has positive lower asymptotic density in the set of all positive odd integers.*

Lemma 1.2. (i) (A. Schinzel) *For each $n = 3, 4, \cdots$, the number $2^{2^n} - 1$ cannot be written as the sum of a prime and two distinct powers of* 2.

(ii) (R. Crocker) *The system consisting of the following* 28 *residue classes*

$$0(3),\ 0(5),\ 1(9),\ 1(10),\ 8(12),\ 8(15),\ 4(18),$$
$$7(20),\ 5(24),\ 29(30),\ 2(36),\ 14(36),\ 17(40),\ 34(45),$$
$$43(45),\ 13(48),\ 37(48),\ 16(60),\ 19(60),\ 26(72),\ 62(72),$$
$$52(90),\ 37(120),\ 49(144),\ 121(144),\ 103(180),\ 106(180),\ 229(360)$$

forms a cover of $\mathbb{Z}$*.*

The proof is not difficult, the following striking result follows from this lemma.

Theorem 1.4 (R. Crocker, Pacific J. Math. 1971)**.** *There are infinitely many positive odd integers not in the form* $p+2^a+2^b$ *where* $a,b\in\mathbb{N}$ *and* p *is a prime.*

Recently I and Mao-Hua Le strengthened Schinzel's result in the following way.

Theorem 1.5 [Z.W. Sun and M.H. Le, Acta Arith., 99(2001), no. 2]**.** *The only solutions of the diophantine equation*

$$2^{2^n}-1=2^a+2^b+p^\alpha \tag{1.11}$$

with $n,a,b,\alpha\in\mathbb{N},\ a>b$ *and* p *being a prime, are as follows:*

$$\begin{aligned}2^{2^2}-1&=2^2+2+3^2=2^3+2^2+3=2^3+2+5,\\ 2^{2^3}-1&=2^3+2^2+3^5=2^7+2+5^3.\end{aligned}$$

Combining our method with that of Crocker, we are near to show that there exist infinitely many positive odd integers not of the form $2^a+2^b+p^\alpha$.

P. Erdős ever conjectured that *every odd positive integer is the sum of a square-free number and a power of* 2.

The following result was obtained by the combined use of covers and the sieve method.

Theorem 1.5 (A. Granville and K. Soundararajan, Ramanujan J., 1998)]**.** *Suppose that every odd positive integer can be written as the sum of a square-free number and a power of* 2*. Then there are infinitely many primes* p *for which* $p^2\nmid 2^{p-1}-1$*. In fact, there then exists a constant* $c>0$ *such that there are arbitrarily large values of* x *for which*

$$|\{\text{primes } p\le x: 2^{p-1}\not\equiv 1 \pmod{p^2}\}|\ge c\pi(x)$$

where $\pi(x)$ denotes the number of primes not more than x.

The following 'almost Goldbach' results are also interesting.

Theorem 1.6. (i) (Yu. V. Linnik, 1950's) Every large even integer can be written a sum of two primes and a bounded number of powers of 2.

(ii) (J.Y. Liu, M.C. Liu and T.Z. Wang, 1998, 2000) Under the Generalized Riemann Hypothesis the upper bound in (i) may be taken as 160.

(iii) (Hong-Ze Li, Acta Arith., 2001) The upper bound in (i) may be taken as 1906.

2. A conjecture of Erdős and its generalizations

For $A = \{a_s(n_s)\}_{s=1}^k$, it can be easily seen that

$$\frac{1}{N}\sum_{x=0}^{N-1} w_A(x) = \sum_{s=1}^{k}\frac{1}{n_s}. \tag{2.1}$$

So, when A forms a cover of $\mathbb{Z}$, we have $\sum_{s=1}^k 1/n_s \geqslant 1$, and equality holds if and only if A covers each integer exactly once.

Soon after his invention of cover, Erdős conjectured that for cover A we have

$$1 < n_1 < \cdots < n_k \Longrightarrow \sum_{s=1}^{k}\frac{1}{n_s} > 1. \tag{2.2}$$

This was confirmed independently by H. Davenport, L. Mirsky, D. Newman and R. Radó.

Theorem 2.1. *Let $A = \{a_s(n_s)\}_{s=1}^k$ be a disjoint cover of $\mathbb{Z}$ with*

$$1 < n_1 \leqslant \cdots \leqslant n_{k-l} < n_{k-l+1} = \cdots = n_k \ (0 < l \leqslant k). \tag{2.3}$$

Then

(i) (Davenport-Mirsky-Newman-Rado) $l \geqslant 2$*;*

(ii) (Š. Znám (1969) & M. Newman (1971)) $l \geqslant p(n_k)$ *where* $p(n_k)$ *is the least prime factor of* n_k.

(iii) (Z.W. Sun, Chin. Quart. J. Math. 1991) $l \geqslant \min_{1\leqslant s\leqslant k-l} \frac{n_k}{(n_s,n_k)}$.

(iv) (Y.G. Chen and Š. Porubský, 1995) *There are* $x_1,\cdots,x_{k-l} \in \mathbb{N}$ *such that*

$$l = \sum_{s=1}^{k-l} \frac{n_s}{(n_s,n_k)} x_s. \tag{2.4}$$

In the 1950s B. H. Neumann investigated covers of general groups by cosets. A nice generalization of Erdős' conjecture is

Herzőg-Schőnheim's Conjecture (Canad. Math. Bull., 1974). *Let* G *be a group and* $G_1,\cdots,G_k$ $(k>1)$ *its subgroups of distinct indices. Then, for any* $a_1,\cdots,a_k \in G$ *system*

$$\mathcal{A} = \{a_sG_s\}_{s=1}^k \tag{2.5}$$

cannot be a disjoint cover (*i.e. a partition*) *of* G.

In 1960's P. Erdős posed the famous unsolved question: *Whether for any arbitrarily large* $c>0$ *there exists a cover* $A=\{a_s(n_s)\}_{s=1}^k$ *of* $\mathbb{Z}$ *with all the moduli* $n_1,\cdots,n_k$ *distinct and greater than* c*?*

Recently we got

Theorem 2.2 [Z. W. Sun, J. Algebra 273(2004), 153–175]**.** *Let* $\{a_iG_i\}_{i=1}^k$ ($k>1$ *and* $n_1=[G:G_1]\leqslant\cdots\leqslant n_k=[G:G_k]$) *be a uniform cover of a group* G *by left cosets* (*i.e. it covers every element of* G *with the same multiplicity*)*. Assume that* $G_1,\cdots,G_k$ *are all subnormal in* G*. Then the indices* $n_1,\cdots,n_k$ *cannot be distinct. If each of the indices occurs at most* $M>1$ *times, then*

$$\log n_1 \leqslant \frac{e^\gamma}{\log 2} M\log^2 M + O(M\log M\log\log M) \tag{2.6}$$

where γ is the Euler constant and the O-constant is absolute.

We emphasize that, for uniform covers of groups by cosets of subnormal subgroups, the above theorem confirms the generalized Herzog–Schőnheim conjecture and answers the analogous question of Erdős negatively.

3. Connections with the linear form $\sum_{s=1}^k x_s/n_s$

Theorem 3.1. (i) [M.Z. Zhang, 1989] *If $A=\{a_s(n_s)\}_{s=1}^k$ forms a cover of $\mathbb{Z}$, then*

$$\sum_{s\in I}\frac{1}{n_s}\in\mathbb{Z}^+ \ for\ some\ I\subseteq\{1,\cdots,k\}. \tag{3.1}$$

(ii) [Z. W. Sun, Trans. Amer. Math. Soc., 1996] *If $A=\{a_s(n_s)\}_{s=1}^k$ forms an m-cover (i.e. $w_A(x)\geqslant m$ for all $x\in\mathbb{Z}$), then for any $m_1,\cdots,m_k\in\mathbb{Z}^+$ there are at least m positive integers in the form $\sum_{s\in I}\frac{m_s}{n_s}$ with $I\subseteq\{1,\cdots,k\}$.*

(iii) [Z. W. Sun, Proc. Amer. Math. Soc., 1999] *Let $A=\{a_s(n_s)\}_{s=1}^k$ form an m-cover and $J\subseteq\{1,\cdots,k\}$. Then for any $m_1,\cdots,m_k\in\mathbb{Z}$ we have*

$$\left|\left\{I\subseteq\{1,\cdots,k\}:\ I\neq J\ \&\ \sum_{s\in I}\frac{m_s}{n_s}-\sum_{s\in J}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\geqslant m. \tag{3.2}$$

Parts (ii) and (iii) are different extensions of part (i).

For $\alpha\in\mathbb{R}$ and $\beta>0$ we let $\alpha+\beta\mathbb{Z}=\{\alpha+\beta x: x\in\mathbb{Z}\}$. Instead of systems of residue classes we may consider a general system in the form

$$\mathcal{A}=\{\alpha_s+\beta_s\mathbb{Z}\}_{s=1}^k. \tag{3.3}$$

By inventing a combined method involving linear algebra, analysis and Stirling numbers, we were able to characterize general covers (not having a fixed covering function) for the first time.

Theorem 3.2 [Z.W. Sun, Acta Arith., **72**(1995), no.2, 109–129]**.** *For system* (3.3) *the following statements are equivalent:*

(a) (3.3) *forms an m-cover of $\mathbb{Z}$.*

(b) (3.3) *covers $|S(\mathcal{A})|$ consecutive integers at least m times where*

$$S(\mathcal{A})=\left\{\left\{\sum_{s\in I}\frac{1}{\beta_s}\right\}:\ I\subseteq\{1,\cdots,k\}\right\}. \tag{3.4}$$

(c) *For any $\theta\in[0,1)$ and $n=0,1,\cdots,m-1$ we have*

$$\sum_{\substack{I\subseteq\{1,\cdots,k\}\\ \{\sum_{s\in I}1/\beta_s\}=\theta}}(-1)^{|I|}\binom{[\sum_{s\in I}1/\beta_s]}{n}e^{2\pi i\sum_{s\in I}\alpha_s/\beta_s}=0. \tag{3.5}$$

Note that $|S(\mathcal{A})|\leqslant 2^k$ depends on those β's! So, that (b) implies (a) gave more detailed information than the following conjecture of Erdős

$$A=\{a_s(n_s)\}_{s=1}^k\ \textit{forms a cover if it covers integers from } 1 \textit{ to } 2^k \tag{3.6}$$

proved by R.B. Crittenden and C.L. Vanden Eynden [Proc. Amer. Math. Soc. **24**(1970)].

The equivalence of (a) and (b) indicates that *the covering function $w_{\mathcal{A}}(x)$ ($x\in\mathbb{Z}$) takes the least value $m(A)=\min_{x\in\mathbb{Z}}w_A(x)$ when x ranges over an interval $[a,a+|S(\mathcal{A})|)$ of length $|S(\mathcal{A})|$.*

That $(a)\Leftrightarrow(c)$ is useful, we derived from it many new properties of the moduli in an m-cover.

Theorem 3.3. *Let $A=\{a_s(n_s)\}$ be an m-cover of $\mathbb{Z}$.*

(i) [Z. W. Sun, Trans. Amer. Math. Soc. 348(1996)] *If*

$$n_1\leqslant\cdots\leqslant n_{k-l}<n_{k-l+1}=\cdots=n_k\ (0\leqslant l<k), \tag{3.7}$$

then

$$l\geqslant\frac{n_k}{n_{k-l}}>1,\quad \textit{or}\ \sum_{s=1}^{k-l}\frac{1}{n_s}\geqslant m\ \textit{and hence}\ \sum_{s=1}^{k}\frac{1}{n_s}\geqslant m+\frac{1}{n_k}>m. \tag{3.8}$$

(ii) [Z. W. Sun, Trans. Amer. Math. Soc. 1996] *If* $\{a_s(n_s)\}_{s\neq t}$ *fails to be an* m*-cover, then for any* $a\in\mathbb{Z}$ *there exist* $I,J\subseteq\{1,\cdots,k\}$ *such that*

$$\frac{a}{n_t}\equiv\sum_{s\in I}\frac{1}{n_s}-\sum_{s\in J}\frac{1}{n_t}\pmod 1. \tag{3.9}$$

(iii) [Z. W. Sun, Proc. Amer. Math. Soc. 127(1999)] *If* A *is a minimal* m*-cover (i.e. no proper subcover of* A *is an* m*-cover again), then for any* $t=1,\cdots,k$ *there exists an* $\alpha_t\in[0,1)$ *such that the set*

$$S_t(A)=\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\}:I\subseteq\{1,\cdots,k\}\setminus\{t\},\ \left[\sum_{s\in I}\frac{1}{n_s}\right]\geqslant m-1\right\} \tag{3.10}$$

contains

$$\left\{\frac{\alpha_t+r}{n_t}:\ r=0,1,\cdots,n_t-1\right\}. \tag{3.11}$$

Here part (i) improves the Davenport-Mirsky-Newman-Rado result which says that if $l=1$ then $\sum_{s=1}^k 1/n_s>1$ (i.e. A is not a disjoint cover), part (ii) shows that m-covers are related to difference sets, part (iii) indicates that if A is a minimal m-cover then for each $t=1,\cdots,k$ the set

$$\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\}:I\subseteq\{1,\cdots,k\}\setminus\{t\}\right\} \tag{3.12}$$

contains an arithmetic progression of length n_t with common difference $1/n_t$.

Conjecture (Z. W. Sun)**.** *Let* $A=\{a_s(n_s)\}_{s=1}^k$ *be a minimal* m*-cover.*

(i) *There exist a chain* $\emptyset\neq I_1\subset\cdots\subset I_m\subseteq\{1,\cdots,k\}$ *such that*

$$\sum_{s\in I_t}\frac{1}{n_s}\in\mathbb{Z}\qquad\text{for all } t=1,\cdots,m. \tag{3.13}$$

(ii) *Set*

$$S(A)=\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\}:I\subseteq\{1,\cdots,k\}\right\}. \tag{3.14}$$

Then

$$S(A) \supseteq \left\{ \frac{r}{d} : r = 0, 1, \cdots, d-1 \right\} \quad \text{if } \frac{1}{d} \in S. \tag{3.15}$$

$A = \{a_s(n_s)\}_{s=1}^k$ is said to be an exact m-cover if $w_A(x) = m$ for all $x \in \mathbb{Z}$. In 1991 Ming-Zhi Zhang showed that an exact m-cover may not have a proper subcover which is an exact n-cover for some $n < m$. Zhi-Wei Sun [Acta Arith., **81**(1997), no. 2, 175–198] characterized exact m-covers in several ways. From the characterizations we deduced some properties of exact m-covers. Here we summarize the central results.

Theorem 3.4 (Z.W. Sun, Acta Arith., 1995, 1997)**.** *Let $A = \{a_s(n_s)\}_{s=1}^k$ be an exact m-cover. Then*

(i) *For any $\emptyset \neq J \subset \{1, \cdots, k\}$,*

$$\sum_{s \in I} \frac{1}{n_s} = \sum_{s \in J} \frac{1}{n_s} \quad \text{for some } I \subseteq \{1, \cdots, k\} \text{ with } I \neq J. \tag{3.16}$$

(ii) *For $a = 0, 1, 2, \cdots$ and $t = 1, \cdots, k$ we have*

$$\left| \left\{ I \subseteq \{1, \cdots, k\} : t \notin I \ \&\ \sum_{s \in I} \frac{1}{n_s} = \frac{a}{n_t} \right\} \right| \geqslant \binom{m-1}{[a/n_t]} \tag{3.17}$$

where the lower bounds are best possible.

(iii) *If $\emptyset \neq I \subseteq \{1, \cdots, k\}$ and $(n_s, n_t) \mid a_s - a_t$ for all $s, t \in I$, then we have*

$$\left\{ \left\{ \sum_{s \in J} \frac{1}{n_s} \right\} : J \subseteq \bar{I} \right\} \supseteq \left\{ \frac{r}{[n_s]_{s \in I}} :\ 0 \leqslant r < [n_s]_{s \in I} \right\} \tag{3.18}$$

where $\bar{I} = \{1, \cdots, k\} \setminus I$ and $[n_s]_{s \in I}$ denotes the least common multiple of those n_s with $s \in I$, moreover for any $r = 0, 1, \cdots, [n_s]_{s \in I} - 1$ we have

$$\left| \left\{ J \subseteq \bar{I} : \left\{ \sum_{s \in J} \frac{1}{n_s} \right\} = \frac{r}{[n_s]_{s \in I}} \right\} \right| \geqslant \frac{\prod_{s \in I} n_s}{[n_s]_{s \in I}}. \tag{3.19}$$

where

(iv) *The number of solutions of the equation*

$$\sum_{s=1}^{k}\frac{x_s}{n_s}=c \quad \text{with } 0\leqslant x_s<n_s \text{ for } s=1,\cdots,k, \tag{3.20}$$

is the sum of finitely many (not necessarily distinct) prime factors of $n_1,\cdots,n_k$ *if* $c\neq 0,1,2,\cdots$, *and at least* $\binom{k-m}{n}$ *if* c *equals a nonnegative integer* n.

The following general results are nice in my opinion.

Theorem 3.5 [Z.W. Sun, Combinatorica, 23(2003), 681–691]**.** *Let* n_0 *be a positive period of the covering function of* $A=\{a_s(n_s)\}_{s=1}^{k}$. *Then we have*

(a) $\{\sum_{s\in J}\frac{1}{n_s}: J\subseteq\{1,\cdots,k-1\}\}\supseteq\{\frac{r}{n_k}: r=0,1,\cdots,\frac{n_k}{(n_0,n_k)}-1\}$.

(b) *If* $n_1\leqslant\cdots\leqslant n_{k-l}<n_{k-l+1}=\cdots=n_k$ $(0<l<k)$, *then for any positive integer* $r<n_k/n_{k-l}$ *with* $r\not\equiv 0 \pmod{n_k/(n_0,n_k)}$, *the binomial coefficient* $\binom{l}{r}$ *can be written as the sum of some (not necessarily distinct) prime divisors of* n_k.

(c) $M(A)=\max_{x\in\mathbb{Z}}w_A(x)$ *can be expressed in the form* $\sum_{s=1}^{k}m_s/n_s$ *where* $m_1,\cdots,m_k\in\mathbb{Z}^+$.