Numerical Validation of Solutions of Variational Inequalities via Poincaré-Miranda Theorem: Theory

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Abstract

The Poincaré-Miranda theorem is used to give a numerical validation of the existence of solution of variational inequality, the validation can be performed on digital computer. Numerical results for the standard test problems will be reported separately.

1 Introduction

Let \( l \in \{ R \cup \{-\infty\}\}^n \) and \( u \in \{ R \cup \{\infty\}\}^n \) be given with \( l < u \). Let \( F : [l, u] \to R^n \) be a given mapping. A box constrained variational inequality problem, denoted by \( VIP(l, u, F) \), is to find an \( x^* \in [l, u] = \{ x \in R^n : l \leq x \leq u \} \) such that

\[
(x - x^*)^T F(x^*) \geq 0 \quad \forall x \in [l, u].
\]  

(1.1)

Such a problem is also called the mixed complementarity problem, see Ferris and Pang [5]. In particular, when \( l_i = 0 \) and \( u_i = +\infty \) for \( i = 1, \ldots, n \), the problem (1.1) reduces to the standard nonlinear complementarity problem, denoted by \( NCP(F) \), which is to find an \( x^* \) such that

\[
x^* \geq 0, \quad F(x^*) \geq 0, \quad (x^*)^T F(x^*) = 0.
\]

(1.2)

It is easy to see that \( x^* \) is a solution of \( VIP(l, u, F) \) if and only if

\[
H(x) = \text{mid}\{x - l, x - u, F(x)\} = 0,
\]

(1.3)

where “mid” is the componentwise median operation. Note that function \( H(x) \) may be not differentiable. For the case of complementarity problem, system (1.3) has the form

\[
H(x) = \min\{x, F(x)\} = 0,
\]

(1.4)

where “min” is taken componentwise.

Plenty of numerical algorithms have been developed for getting an approximate solution of (1.1) by solving (1.3). See Facchinei and Pang [3, pp.793–876] for a comprehensive treatment. Of course in practical settings, without any further preinformation we have to decide whether there exists a solution at all, this can usually be done via computationally testing the conditions of some existence theorems. We call a positive test of the existence of a solution of the problem

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VIP\((l,u,F)\) as a numerical validation. Let \(\hat{x} = (\hat{x}_i) \in \mathbb{R}^n\) and let \(r = (r_i) \in \mathbb{R}_{+}^n\), they could be respectively an output of a numerical algorithm and a guess of its error. The validation is usually performed over an interval vector

\[
[x] = [\hat{x} - r, \hat{x} + r] = \{x \in \mathbb{R}^n : \hat{x} - r \leq x \leq \hat{x} + r\}. \tag{1.5}
\]

Validation of the existence of a solution is extremely important for delivering a reliable approximate solution. We know, an iterative algorithm stops to output an iterate where the value of a merit function is small enough. However, a small value of the merit function does not mean that there must be a solution \(x^*\) near the iterate. On the other hand, the convergence of the sequence of iterates requires in the most case certain properties of the unknown solution, which can not be tested for proving the existence of a solution. Without it, an output of an algorithm may be of doubtful utility. We demonstrate this by the following example.

**Example 1.1** Consider the problem VIP\((l,u,F)\), where

\[
l = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad F(x) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}x + \begin{pmatrix} 2 \\ 1 \\ -10^{-6} \end{pmatrix}.
\]

Let

\[
x^0 = \begin{pmatrix} 0 \\ 0 \\ 10^{-6} \end{pmatrix}.
\]

In the semismooth Newton’s method by Kanzow and Fukushima [6], the condition

\[
g_{\alpha\beta}(x) < \varepsilon
\]

is used as the stopping criterion, where \(g_{\alpha\beta}(x)\) is the so-called gap-function. In the implementation setting of [6], \(\varepsilon = 10^{-5}\) was chosen. Here we compute \(g_{\alpha\beta}(x^0) \approx 4 \times 10^{-13}\). However, the problem VIP\((l,u,F)\) has actually no solution.

So far there is a very few validation methods for variational inequalities [?] and complementarity problems [1]. The motivation of the present work is to design a new validation method by making use of the Poincaré-Miranda theorem.

**Theorem 1.1** (Poincaré-Miranda, [10]) Let \(x = [\hat{x} - r, \hat{x} + r]\) be defined by (1.5), define its \(n\) pairs of parallel and opposite faces

\[
[x]^+_i = \{x \in [x] : x_i = \hat{x}_i + r_i\}, \tag{1.6}
\]

\[
[x]^-_i = \{x \in [x] : x_i = \hat{x}_i - r_i\}, \tag{1.7}
\]

where \(i = 1, \ldots, n\). Let \(G : [x] \to \mathbb{R}^n\) be continuous. Then the system \(G(x) = 0\) has a solution in \([x]\) if for any \(i = 1, \ldots, n\), we have

\[
G_i(x) \begin{cases} 
\leq 0 & \text{if } x \in [x]^+_i, \\
\geq 0 & \text{if } x \in [x]^-_i.
\end{cases} \tag{1.8}
\]
Note that the condition (1.8) of Theorem 1.1 is very hard, if not completely impossible, to be tested. This is why the theorem is never been practically applied to the variational inequalities. In [11] an approach was proposed for testing (1.8) for differentiable systems, which, unfortunately, can not be applied here. In this article we apply Theorem 1.1 to the system

\[ A^{-1} \text{mid} \{x - l, x - u, F(x)\} =: G(x) = 0 \]  

(1.9)

where the matrix \( A^{-1} \) is introduced for preconditioning. Obvious is that, if intervals \( G_i([x]^-) \) and \( G_i([x]^+) \) satisfying

\[ G_i(x) \begin{cases} \in G_i([x]^+) & \text{if } x \in [x]^+ \\ \in G_i([x]^-) & \text{if } x \in [x]^- \end{cases} \]  

(1.10)

are available, then (1.8) follows from the condition

\[ \sup \{ y : y \in G_i([x]^-) \} \leq 0 \leq \inf \{ y : y \in G_i([x]^+) \}, \]  

(1.11)

which ensures the existence of a solution \( x^* \in [x] \) of the system (1.9), and therefore a solution of \( VIP(l, u, F) \). We give validation methods by computing intervals \( G_i([x]^-) \) and \( G_i([x]^+) \) and testing (1.11) for them. Our validation method is new, and will be shown more general to apply than those given in [1].

Meanwhile, as a direct consequence of the existence of the solution \( x^* \in [x] \), we have an error bound

\[ |x^* - \hat{x}| \leq r. \]  

(1.12)

Clearly, a more general validation can deliver a more accurate error bound. Error bound plays a significant role in both numerical computation and theoretical analysis. It has been extensively studied for the linear complementarity problems (see [2, 8, 9, 13]), however, remains open for variational inequality problems. We use the validation method to give error bounds for variational inequalities, numerical results show that our error bounds could be better in several orders of magnitude than those existing.

This article is organized as follows. We present in Sect.3 the validation methods and prove that they are more general to apply than the existing. In Sect.4 we give error bounds by using an established validation method, and show that they are sharper than those existing, an index identification property of the bounds is also presented therein. We perform numerical experiments to demonstrate the efficiency of our validation methods and the accuracy of the error bounds. Some concluding remarks are given in the last section.

2 Notations and Preliminaries

Introduce notations on interval computation. Let \( \underline{A} = (a_{ij}), \; \overline{A} = (\overline{a}_{ij}) \in R^{n \times n} \) with \( a_{ij} \leq \overline{a}_{ij} \), \( i, j = 1, \ldots, n \). Denote by \([A] = [\underline{A}, \overline{A}]\) an interval matrix, which is a set

\[ [A] := \{ A = (a_{ij}) \in R^{n \times n} : \; a_{ij} \leq a_{ij} \leq \overline{a}_{ij} \}. \]

The \((i,j)\)-th element of \([A]\) is denoted by \([a_{ij}]\). Define \(|[A]| := (\max\{|a_{ij}|, |\overline{a}_{ij}|\}) \in R^{n \times n} \). Let \( \underline{x}, \overline{x} \in R^n \) with \( \underline{x} \leq \overline{x} \). An interval vector \([x] = [\underline{x}, \overline{x}]\) is an \( n \times 1 \) real interval matrix, we denote its \( i \)-th component by \([x]_i = [\underline{x}_i, \overline{x}_i]\).
In order to test the condition (1.11), we need computing intervals \(G_i([x]^-)\) and \(G_i([x]^+)\). One way for achieving this is by using slope. Let \(\hat{x}\) and \(x\) be given vectors. A slope of \(G\) associated with \(\hat{x}\) and \(x\), denoted by \(\delta G(\hat{x}, x)\), is a matrix such that

\[
G(\hat{x}) = G(x) + \delta G(\hat{x}, x)(\hat{x} - x).
\]

(2.13)

Let \([x]\) be a given interval vector and let \(\hat{x} \in [x]\) be a given fixed vector. \(G\) has a slope when it is Lipschitz continuous. See [12], for example. An interval extension of a slope of the mapping \(G\) associated with \(\hat{x}\) and \([x]\), denoted by \(\delta G(\hat{x}, [x])\), is a matrix of interval entries such that for any \(x \in [x]\)

\[
\delta G(\hat{x}, x) \in \delta G(\hat{x}, [x]).
\]

(2.14)

Here in we call \(\delta G(\hat{x}, [x])\) interval slope for short. Clearly we have

\[
G(x) \in G(\hat{x}) + \delta G(\hat{x}, [x])([x] - \hat{x}).
\]

(2.15)

3 Computation of Interval Slope

In this section we study computing an interval slope \(\delta H(\hat{x}, [x])\) of the mapping \(H\) defined by (1.3). First of all we give a slope \(\delta H(\hat{x}, x)\) of the following form.

**Theorem 3.1** Let \(\hat{x}, x \in \mathbb{R}^n\) be given vectors, let \(\hat{y} = (\hat{y}_i) = \hat{x} - F(\hat{x})\) and \(y = (y_i) = x - F(x)\). Let \(\delta F(\hat{x}, x)\) be a slope of \(F\) associated with \(\hat{x}\) and \(x\), denote its \(i\)-th row by \(\delta F_i(\hat{x}, x)\). Then the mapping \(H(x) = \text{mid}\{x - l, x - u, F(x)\}\) has a slope \(\delta H(\hat{x}, x) \in \mathbb{R}^{n \times n}\), whose the \(i\)-th row vector \(\delta H_i(\hat{x}, x) \in \mathbb{R}^n\) has the form:

\[
\delta H_i(\hat{x}, x) = e_i^T - \alpha_i(e_i^T - \delta F_i(\hat{x}, x)),
\]

(3.16)

where \(e_i\) denotes the \(i\)-th unit coordinate vector, and the coefficient \(\alpha_i \in [0, 1]\) is given by

\[
\alpha_i = \begin{cases} 
0 & \text{if } y_i, \hat{y}_i \leq l_i \text{ or } y_i, \hat{y}_i \geq u_i \\
1 & \text{if } y_i, \hat{y}_i \in (l_i, u_i) \\
\frac{\text{mid}\{l_i, u_i, y_i\} - \text{mid}\{l_i, u_i, \hat{y}_i\}}{y_i - \hat{y}_i} & \text{otherwise.}
\end{cases}
\]

(3.17)

**Proof:** We prove \(H_i(x) - H_i(\hat{x}) = \delta H_i(\hat{x}, x)(x - \hat{x})\) for the three cases.

(1) If \(\hat{y}_i, y_i \leq l_i\), then we have \(F_i(x) \geq x_i - l_i, F_i(\hat{x}) \geq \hat{x}_i - l_i\), these yield \(H_i(x) = x_i - l_i\) and \(H_i(\hat{x}) = \hat{x}_i - l_i\). Note that (3.16) reads \(\delta H_i(\hat{x}, x) = e_i\) and (3.17) reads \(\alpha_i = 0\). So

\[
H_i(x) - H_i(\hat{x}) = (x_i - l_i) - (\hat{x}_i - l_i) = e_i^T(x - \hat{x}) = \delta H_i(\hat{x}, x)(x - \hat{x}).
\]

For the case of \(\hat{y}_i, y_i \geq u_i\), the same argument can be derived in a similar way. We omit it.

(2) If \(\hat{y}_i, y_i \in (l_i, u_i)\), then we have \(F_i(\hat{x}), F_i(x) \in (x_i - u_i, x_i - l_i),\) this yields \(H_i(\hat{x}) = F_i(\hat{x})\) and \(H_i(x) = F_i(x)\). Note that (3.16) reads \(\delta H_i(\hat{x}, x) = \delta F_i(\hat{x}, x)\) and (3.17) reads \(\alpha_i = 1\). So

\[
H_i(x) - H_i(\hat{x}) = F_i(x) - F_i(\hat{x}) = \delta F_i(\hat{x}, x)(x - \hat{x}) = \delta H_i(\hat{x}, x)(x - \hat{x}).
\]

(3) Otherwise, if it is not the above two cases, we have \(\hat{y}_i \neq y_i\). From the definition of the slope it follows that

\[
y_i - \hat{y}_i = (x_i - F_i(x)) - (\hat{x}_i - F_i(\hat{x})) = (e_i^T - \delta F_i(\hat{x}, x))(x - \hat{x}).
\]
This, together with (3.16) and (3.17), yields

\[
H_i(x) - H_i(\hat{x}) = (x_i - \hat{x}_i) - (\text{mid}\{l_i, u_i, y_i\} - \text{mid}\{l_i, u_i, \hat{y}_i\}) \\
= e_i^T(x - \hat{x}) - \alpha_i(y_i - \hat{y}_i) \\
= (e_i^T - \alpha_i(e_i^T - \delta F_i(\hat{x}, x)))(x - \hat{x}) \\
= \delta H_i(\hat{x}, x)(x - \hat{x}).
\]

The relation \(\alpha_i \in [0, 1]\) follows from the fact

\[
|\text{mid}\{l_i, u_i, y_i\} - \text{mid}\{l_i, u_i, \hat{y}_i\}| \leq |y_i - \hat{y}_i|
\]

since \(\text{mid}\{l_i, u_i, \cdot\}\) is actually the projection onto \([l_i, u_i]\). This completes the proof.

Remember that a matrix \(A\) is called a P-matrix if for any vector \(x \in \mathbb{R}^n\) with \(x \neq 0\) we always have

\[
\max_i x_i(Ax)_i > 0.
\]

A P-matrix is nonsingular. We have the following property of the slope \(\delta H(\hat{x}, x)\) given in Theorem 3.1.

**Theorem 3.2** In the setting of Theorem 3.1, if \(\delta F(\hat{x}, x)\) is a P-matrix, then \(\delta H(\hat{x}, x)\) is also a P-matrix, and so is nonsingular.

**Proof:** Let \(\Lambda = \text{diag}(\alpha_i) \in \mathbb{R}^{n \times n}\), where \(\alpha_i \in [0, 1]\) is given in (3.17). From (3.16) it follows

\[
\delta H(\hat{x}, x) = I - \Lambda + \Lambda \delta F(\hat{x}, x).
\]

Let \(z \in \mathbb{R}^n\) be non-zero. Since \(\delta F(\hat{x}, x)\) is a P-matrix, we have

\[
z_k(\delta F(\hat{x}, x)z)_k = \max_i z_i(\delta F(\hat{x}, x)z)_i > 0.
\]

It is clear that \(z_k \neq 0\). For the case of \(\alpha_k = 0\), we have

\[
z_k(\delta H(\hat{x}, x)z)_k = z_k^2 > 0.
\]

For the case of \(\alpha_k \neq 0\), then \(\alpha_k > 0\). We have

\[
z_k(\delta H(\hat{x}, x)z)_k = (1 - \alpha_k)z_k^2 + \alpha_k z_k(\delta F(\hat{x}, x)z)_k \geq \lambda_k z_k(\delta F(\hat{x}, x)z)_k > 0.
\]

Both of the two cases yield that

\[
\max_i z_i(\delta H(\hat{x}, x)z)_i \geq z_k(\delta H(\hat{x}, x)z)_k > 0,
\]

which means that \(\delta H(\hat{x}, x)\) is a P-matrix. This completes the proof.

Now we give an interval slope of the mapping \(H\) defined in (1.3).

**Theorem 3.3** Let \(\hat{x} \in \mathbb{R}^n\) be a given vector and let \([x]\) be a given interval vector, let \(\hat{y}_i = \hat{x}_i - F_i(\hat{x})\), and let \(\underline{y}_i\) and \(\overline{y}_i\) be the values such that for \(i = 1, \ldots, n\)

\[
\{x_i - F_i(x) : x \in [x]\} \subseteq [\underline{y}_i, \overline{y}_i].
\]
Let $\delta F(\hat{x}, [x])$ be an interval slope of $F$ associated with $\hat{x}$ and $[x]$, denote its $i$-th row by $\delta F_i(\hat{x}, [x])$. Then the mapping $H(x) = \text{mid}\{x - l, x - u, F(x)\}$ has an interval slope $\delta H(\hat{x}, [x])$, whose $i$-th row vector $\delta H_i(\hat{x}, [x])$ (an interval vector) has the form:

$$
\delta H_i(\hat{x}, [x]) = e_i - [\alpha_i](e_i - \delta F_i(\hat{x}, [x])), \quad (3.19)
$$

where $e_i$ denotes the $i$-th unit coordinate vector, and $[\alpha_i]$ is an interval defined by

$$
[\alpha_i] = \begin{cases}
[0, 0] & \text{if } y_i \leq \hat{y}_i \leq \bar{y}_i \leq l_i \\
[0, \frac{\bar{y}_i - l_i}{\bar{y}_i - \hat{y}_i}] & \text{if } y_i \leq \hat{y}_i \leq l_i < \bar{y}_i < u_i \\
[\frac{\hat{y}_i - l_i}{\bar{y}_i - \hat{y}_i}, 1] & \text{if } y_i \leq l_i < \hat{y}_i \leq \bar{y}_i < u_i \\
[0, \frac{u_i - l_i}{u_i - y_i}] & \text{if } y_i \leq l_i < u_i < \hat{y}_i < u_i \\
\min\left\{\frac{\hat{y}_i - l_i}{\bar{y}_i - \hat{y}_i}, \frac{u_i - \hat{y}_i}{u_i - y_i}\right\}, 1 & \text{if } y_i \leq l_i < \hat{y}_i < u_i \leq \bar{y}_i \\
[0, \frac{u_i - l_i}{\bar{y}_i - \hat{y}_i}] & \text{if } l_i < y_i \leq \hat{y}_i \leq \bar{y}_i < u_i \\
[1, 1] & \text{if } l_i < y_i \leq \hat{y}_i \leq \bar{y}_i < u_i \\
[\frac{u_i - \hat{y}_i}{\bar{y}_i - \hat{y}_i}, 1] & \text{if } l_i < y_i \leq \hat{y}_i < u_i \leq \bar{y}_i \\
[0, \frac{u_i - y_i}{\bar{y}_i - y_i}] & \text{if } l_i < y_i < u_i \leq \hat{y}_i \leq \bar{y}_i \\
[0, 0] & \text{if } l_i < u_i \leq y_i \leq \bar{y}_i \leq \hat{y}_i \leq u_i \leq \bar{y}_i
\end{cases}
\quad (3.20)
$$

for the case $-\infty < l_i < u_i < \infty$; and defined by

$$
[\alpha_i] = \begin{cases}
[1, 1] & \text{if } l_i < y_i \leq \hat{y}_i \leq \bar{y}_i < u_i \\
[\frac{u_i - \hat{y}_i}{\bar{y}_i - \hat{y}_i}, 1] & \text{if } l_i < y_i \leq \hat{y}_i < u_i \leq \bar{y}_i \\
[0, \frac{u_i - y_i}{\bar{y}_i - y_i}] & \text{if } l_i < y_i < u_i \leq \hat{y}_i \leq \bar{y}_i \\
[0, 0] & \text{if } l_i < u_i \leq y_i \leq \bar{y}_i \leq \hat{y}_i \leq u_i \leq \bar{y}_i
\end{cases}
\quad (3.21)
$$
for the case $-\infty < l_i < u_i < \infty$; and defined by

$$[\alpha_i] = \begin{cases} 
[0,0] & \text{if } y_i \leq \hat{y}_i \leq \underline{y}_i \leq l_i \\
[0, \frac{\underline{y}_i - l_i}{\bar{y}_i - \hat{y}_i}] & \text{if } y_i \leq \hat{y}_i \leq l_i < \underline{y}_i \\
[\frac{\bar{y}_i - l_i}{\bar{y}_i - \hat{y}_i}, 1] & \text{if } y_i \leq l_i < \hat{y}_i \leq \underline{y}_i \\
[1,1] & \text{if } l_i < y_i \leq \hat{y}_i \leq \underline{y}_i 
\end{cases}$$

(3.22)

for the case $-\infty < l_i < u_i = \infty$; and defined by

$$[\alpha_i] = [1,1]$$

(3.23)

for the case $-\infty = l_i < u_i = \infty$. Moreover we have $[\alpha_i] \subseteq [0,1]$.

**Proof:** From Theorem 3.1 it follows that $\delta H_i(\hat{x}, x) = e_i^T - \alpha_i(e_i^T - \delta F_i(\hat{x}, x))$, where $\alpha_i$ is given by (3.17). It is sufficient to prove that $\alpha_i \in [\alpha_i]$ for any $x \in [x]$, this yields $\delta H_i(\hat{x}, x) \in \delta H_i(\hat{x}, [x])$, and therefore the conclusion of the theorem. We divide the proof into four cases.

1. For the case of $y_i \leq \bar{y}_i \leq \underline{y}_i \leq l_i$, from (3.17) it follows that $\alpha_i = 0 \in [\alpha_i]$.

2. For the case of $y_i \leq \bar{y}_i \leq l_i < \underline{y}_i < u_i$ (the second case (3.20)), since we have for any $x \in [x]$ that $y_i = x_i - F_i(x) \leq l_i$ or $y_i = x_i - F_i(x) \in (l_i, u_i)$, from (3.17) it follows that $\alpha_i = 0$ when $y_i \leq l_i$, and

$$\alpha_i = \frac{y_i - l_i}{\bar{y}_i - \hat{y}_i} \leq \frac{\bar{y}_i - l_i}{\bar{y}_i - \hat{y}_i}$$

when $y_i \in (l_i, u_i)$. This yields

$$\alpha_i \in \left[0, \frac{\bar{y}_i - l_i}{\bar{y}_i - \hat{y}_i}\right] = [\alpha_i].$$

3. For the case of $y_i \leq l_i < \bar{y}_i < \underline{y}_i < u_i$ (the third case in (3.20)), since we have for any $x \in [x]$ that $y_i = x_i - F_i(x) \leq l_i$ or $y_i = x_i - F_i(x) \in (l_i, u_i)$, from (3.17) it follows that $\alpha_i = 1$ when $y_i \in (l_i, u_i)$, and

$$\alpha_i = \frac{\hat{y}_i - l_i}{\bar{y}_i - \hat{y}_i} \geq \frac{\bar{y}_i - l_i}{\bar{y}_i - \hat{y}_i}$$

when $y_i \leq l_i$. This yields

$$\alpha_i \in \left[\frac{\hat{y}_i - l_i}{\bar{y}_i - \hat{y}_i}, 1\right] = [\alpha_i].$$

4. For the case of $y_i \leq l_i < \bar{y}_i < u_i \leq \underline{y}_i$ (the fifth case in (3.20)), since we have for any $x \in [x]$ that $y_i = x_i - F_i(x) \leq l_i$ or $y_i = x_i - F_i(x) \in (l_i, u_i)$ or $y_i = x_i - F_i(x) \geq u_i$, from (3.17) it follows that $\alpha_i = 1$ when $y_i \in (l_i, u_i)$, and

$$\alpha_i = \frac{\hat{y}_i - l_i}{\bar{y}_i - \hat{y}_i} \geq \frac{\bar{y}_i - l_i}{\bar{y}_i - \hat{y}_i}$$

when $y_i \leq l_i$, and

$$\alpha_i = \frac{u_i - \hat{y}_i}{\bar{y}_i - \hat{y}_i} \geq \frac{u_i - \hat{y}_i}{\bar{y}_i - \hat{y}_i}$$

when $\bar{y}_i \leq \hat{y}_i \leq u_i$.
when \( y_i \geq u_i \). This yields
\[
\alpha_i \in \left[ \min \left\{ \frac{\hat{y}_i - l_i}{\hat{y}_i - \hat{y}_i}, \frac{u_i - \hat{y}_i}{\hat{y}_i - \hat{y}_i} \right\}, 1 \right] = [\alpha_i].
\]

The proof for the rest cases: the fourth, sixth–tenth cases in (3.20) and (3.21)–(3.23) can be respectively given in the very similar way as above. We omit them. The proof is completed.

**Theorem 3.4** In the setting of Theorem 3.3, if each matrix in the interval slope \( \delta F(\hat{x}, [x]) \) is a P-matrix, then each matrix in the interval slope \( \delta H(\hat{x}, x) \), given in Theorem 3.3, is also a P-matrix, and so is nonsingular.

**Proof:** The proof can be given in a very similar manner to that for Theorem 3.2.

For the special case with \( l_i = 0 \) and \( u_i = \infty \) for \( i = 1, \ldots, n \), the problem \( V I(l, u, F) \) reduces into the nonlinear complementarity problem \( NCP(F) \), and the mapping defined in (1.3) has the form (1.4). In this case, we have the following result.

**Corollary 3.1** In the setting of Theorem 3.3, the mapping \( H(x) = \min \{x, F(x)\} \) has an interval slope \( \delta H(\hat{x}, [x]) \), where \( \delta H_i(\hat{x}, [x]) \) has the form (3.19), and
\[
[\alpha_i] = \begin{cases} 
[0, 0] & \text{if } y_i \leq \hat{y}_i \leq y_i \\
[0, \frac{\hat{y}_i}{\hat{y}_i - y_i}] & \text{if } y_i \leq \hat{y}_i \leq 0 < y_i \\
[\frac{\hat{y}_i}{\hat{y}_i - y_i}, 1] & \text{if } 0 < \hat{y}_i \leq y_i \\
[1, 1] & \text{if } 0 < y_i \leq \hat{y}_i \leq y_i.
\end{cases}
\] (3.24)

Moreover, we have \([\alpha_i] \subseteq [0, 1]\).

**Remark 3.1** The interval slope \( \delta H(\hat{x}, [x]) \) given in the corollary has a similar form to that given by Alefeld et al. in ([1]). However, theirs needs solving optimization problems that is not trivial, while it is not the case of this corollary.

### 4 Numerical Validation

The following is a naive realization of the numerical validation of the conditions of Poincaré-Miranda Theorem for the system \( H(x) = 0 \) defined by (1.3).

**Theorem 4.1** Let \( \hat{x} = (\hat{x}_i) \in R^n \) and \( r = (r_i) \in R^n_+ \) be given. For \( [x] = [\hat{x} - r, \hat{x} + r] \) we let \([x]_i^+ \) and \([x]_i^- \) be defined by (1.6) and (1.7) respectively:
\[
[x]_i^+ = \{ x \in [x] : x_i = \hat{x}_i + r_i \},
[x]_i^- = \{ x \in [x] : x_i = \hat{x}_i - r_i \}.
\]
Let \( A \in \mathbb{R}^{n \times n} \) be non-singular. Let the mapping \( H \) be defined by (1.3) or by (1.4) for the special case of complementarity problem, and let \( \delta H(\hat{x} + r_i e_i, [x]_i^+) \) and \( \delta H(\hat{x} - r_i e_i, [x]_i^-) \) be computed by (3.19)–(3.23). Let
\[
G_i([x]_i^+) := \left[ A^{-1} \left( H(\hat{x} + r_i e_i) + \delta H(\hat{x} + r_i e_i, [x]_i^+) ([x]_i^+ - \hat{x} - r_i e_i) \right) \right]_i \quad (4.25)
\]
and
\[
G_i([x]_i^-) := \left[ A^{-1} \left( H(\hat{x} - r_i e_i) + \delta H(\hat{x} - r_i e_i, [x]_i^-) ([x]_i^- - \hat{x} + r_i e_i) \right) \right]_i. \quad (4.26)
\]
If for each \( i = 1, \ldots, n \) either
\[
\sup \left\{ y : y \in G_i([x]_i^-) \right\} \leq 0 \leq \inf \left\{ y : y \in G_i([x]_i^+) \right\} \quad (4.27)
\]
or
\[
\sup \left\{ y : y \in G_i([x]_i^+) \right\} \leq 0 \leq \inf \left\{ y : y \in G_i([x]_i^-) \right\} \quad (4.28)
\]
holds, then there is a solution of \( H(x) = 0 \), and therefore a solution of \( VI(l, u, F) \) in the interval vector \([x]\).

**Proof:** It is a direct consequence of Miranda theorem and Theorem 3.3.

Here we have to mention that by using Krawczyk operator we can also give a numerical validation of solution of \( VI(l, u, F) \).

**Theorem 4.2** Let \( \hat{x} = (\hat{x}_i) \in \mathbb{R}^n \) and \( r = (r_i) \in \mathbb{R}^n_+ \) be given. For \([x] = [\hat{x} - r, \hat{x} + r]\) Let \( A \in \mathbb{R}^{n \times n} \) be non-singular. Let the mapping \( H \) be defined by (1.3) or by (1.4) for the special case of complementarity problem, and let \( \delta H(\hat{x}, [x]) \) be computed by (3.19)–(3.23). If
\[
\hat{x} - A^{-1} H(x) + (I - A^{-1}) \delta H(\hat{x}, [x])([x] - \hat{x}) =: K(\hat{x}, [x], A) \subseteq [x], \quad (4.29)
\]
then there is a solution of \( H(x) = 0 \), and therefore a solution of \( VI(l, u, F) \) in the interval \([x]\): if
\[
K(\hat{x}, [x], A) \cap [x] = \emptyset
\]
then there is no solution of \( VI(l, u, F) \) in the interval \([x]\).

**Proof:** The proof is very similar to that given in [1].

Theorem 4.2 is actually to test the condition of Brouwer fixed point theorem. It is interesting that Theorem 4.1 can be shown in a certain sense more general to apply than the Theorem 4.2 although the Poincaré-Miranda theorem was proven equivalent to the Brouwer fixed point theorem [4, 7]. We combine the two theorems to give the following numerical validation method.

**Algorithm 4.1** Let \( \hat{x} = (\hat{x}_i) \in \mathbb{R}^n \) and \( r = (r_i) \in \mathbb{R}^n_+ \) be given. For \([x] = [\hat{x} - r, \hat{x} + r]\) we let \([x]_i^+\) and \([x]_i^-\) be defined by
\[
[x]_i^+ = \{ x \in [x] : x_i = \hat{x}_i + r_i \},
\]
\[
[x]_i^- = \{ x \in [x] : x_i = \hat{x}_i - r_i \}.
\]
Let \( A \in \mathbb{R}^{n \times n} \) be non-singular. Let the mapping \( H \) be defined by (1.3) or by (1.4) for the special case of complementarity problem, and let \( \delta H(\hat{x} + r_i e_i, [x]_i^+) \) and \( \delta H(\hat{x} - r_i e_i, [x]_i^-) \) be computed by (3.19)–(3.23).
Compute \( G_i([x]_i^+) = [G_i^+ , G_i^+] \) and \( G_i([x]_i^-) = [G_i^- , G_i^-] \) be computed by (4.25) and (4.26). If for each \( i = 1, \ldots , n \) either

\[
\sup \left \{ y : y \in G_i([x]_i^-) \right \} = G_i^- \leq 0 \leq G_i^+ = \inf \left \{ y : y \in G_i([x]_i^+) \right \}
\]

or

\[
\sup \left \{ y : y \in G_i([x]_i^+) \right \} = G_i^+ \leq 0 \leq G_i^- = \inf \left \{ y : y \in G_i([x]_i^-) \right \}
\]

holds, then there is a solution of \( H(x) = 0 \), and therefore a solution of \( VI(l, u, F) \) in the interval vector \([x]_i\).

Otherwise compute \( K(\hat{x}, [x], A) \) by (4.29). If

\[
K(\hat{x}, [x], A) \cap [x] = \emptyset
\]

then there is no solution of \( VI(l, u, F) \) in the interval \([x]_i\).

**Remark 4.2** Since the real computation is carried out on the floating number system, actually we have to use certain floating number representation of vectors \( l \) and \( u \) in (3.19)–(3.23). If there is overflow for \( l_i \) and/or for \( u_i \), then we set \( l_i = -\infty \) and/or \( u_i = \infty \) respectively, and use (3.21) or (3.22), or use (3.23) to compute the interval slope of \( H \). Otherwise, let \( l_i \leq l_i \leq \bar{l}_i \) and \( \underline{u}_i \leq u_i \leq \bar{u}_i \), where \( l_i, \bar{l}_i, \underline{u}_i \) and \( \bar{u}_i \) are machine number which are known. Then for testing the existence of a solution of \( VI(l, u, F) \), we use in (3.20) \( l_i \) and \( \bar{l}_i \), instead of \( l_i \) and \( u_i \) respectively, to compute the interval slope of \( H \). Here the real treated problem is \( VI(l, \bar{u}, F) \), whose solution must be a solution of \( VI(l, u, F) \) as the domain \([l, \bar{u}]_i\) includes that of the problem \( VI(l, u, F) \). While for testing the non-existence of a solution of \( VI(l, u, F) \), we use in (3.20) \( \bar{l}_i \) and \( \underline{u}_i \) instead of \( l_i \) and \( u_i \) respectively, where the real treated problem is \( VI(l, u, F) \), whose non-existence of solution must imply that the original problem \( VI(l, u, F) \) has no solution in the interval \([x]_i\).

**Remark 4.3** For computing the interval slope of \( H \), it is needed to compute the range \([y_i, \bar{y}_i]_i\), this can be readily done by existing software, e.g. Intlab [14], which is a toolbox build on top of MATLAB for tracking roundoff error.

**References**


