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ON \((m, n)\)-INJECTIVITY OF MODULES

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ABSTRACT

Let \(R\) be a ring. For two fixed positive integers \(m\) and \(n\), a right \(R\)-module \(M\) is called \((m, n)\)-injective if every right \(R\)-homomorphism from an \(n\)-generated submodule of \(R^m\) to \(M\) extends to one from \(R^m\) to \(M\). This definition unifies several definitions on generalizations of injectivity of modules. The aim of this paper is to investigate properties of the \((m, n)\)-injective modules. Various results are developed, many extending known results.
1. INTRODUCTION

Throughout $R$ is an associative ring with identity and all modules are unitary. We write $M_R$ ($rM$) to indicate a right (left) $R$-module, and we use the notation $R_{m \times n}$ for the set of all $m \times n$ matrices over $R$. For $A \in R_{m \times n}$, $A^T$ will denote the transpose of $A$. In general, for an $R$-module $N$, we write $N_{m \times n}$ for the set of all formal $m \times n$ matrices whose entries are elements of $N$. Let $M_R$ and $rN$ be $R$-modules. For $x \in M_{l \times m}$, $s \in R_{m \times n}$ and $y \in N_{n \times k}$, under the usual multiplication of matrices, $xs$ (resp. $xy$) is a well-defined element in $M_{l \times n}$ (resp. $N_{m \times k}$). If $X \subseteq M_{l \times m}$, $S \subseteq R_{m \times n}$ and $Y \subseteq N_{n \times k}$, define

$$l_{M^{l \times m}}(S) = \{ u \in M^{l \times m} : us = 0, \forall s \in S \}$$

$$r_{N^{n \times k}}(S) = \{ v \in N^{n \times k} : sv = 0, \forall s \in S \}$$

$$r_{R^{m \times n}}(X) = \{ s \in R^{m \times n} : xs = 0, \forall x \in X \}$$

$$l_{R^{m \times n}}(Y) = \{ s \in R^{m \times n} : sy = 0, \forall y \in Y \}.$$ 

We will write $N^n = N^{1 \times n}$, $N_n = N^{n \times 1}$, $R^n = R^{1 \times n}$ and $R_n = R^{n \times 1}$. Multiplication maps $x \mapsto ax$ and $x \mapsto xa$ will be denoted $\cdot a$- and $a \cdot$- respectively.

Generalizations of injectivity have been discussed in many papers, for example, see [2, 4, 5, 6, 8, 9, 11, 12, 13, 14, 15, 18]. In this paper, for two fixed positive integers $m$ and $n$, $(m,n)$-injective modules are defined and studied. We prove that $M_R$ is $(m,n)$-injective if and only if $l_{M^{l \times m}}r_{R_{m \times n}}(x_1, x_2, \ldots, x_m) = Mx_1 + Mx_2 + \cdots + Mx_m$ for all $x_i \in R^n$, $i = 1, 2, \ldots, m$ [Theorem 2.4]. This fact is then used to prove that a left Kasch left $(n, m + 1)$-injective ring $R$ is right $(m, n)$-injective [Theorem 2.7]. The $(m,n)$-injective modules are also characterized as those $(m,1)$-injective modules $M_R$ for which $l_{M^n}(I \cap K) = l_{M^n}(I) + l_{M^n}(K)$, where $I$ and $K$ are submodules of $(R_m)_R$ such that $I + K$ is $n$-generated [Theorem 2.9]. Any left Kasch, left $P$-injective and left $IN$-ring $R$ is proved to be right $j$-injective and left dual (i.e., every left ideal of $R$ is a left annihilator) [Theorem 2.13]. Another characterization of $(m,n)$-injective modules is obtained as stated as follows: $M_R$ is $(m,n)$-injective if and only if, for any $z = (m_1, m_2, \ldots, m_n) \in M^n$ and $A \in R_{m \times n}$ satisfying $r_{R_{m \times n}}(A) \subseteq r_{R_{n \times n}}(z)$, $z = yA$ for some $y \in m^n$ [Theorem 2.15]. We use this theorem to prove that $R$ is right $(m,n)$-injective if and only if the exactness of $R^n \to R^n \to R \to 0$ implies the torsionlessness of $N$ [Theorem 2.17] and that $R$ is right $(m,n)$-injective and left $(n,m)$-injective if and only if $R$ is right $(m,n)$-wlec and left $(n,m)$-wlec [Theorem 2.20]. Some
known results appearing in [2, 6, 8, 10, 12, 14, 15] are obtained as corollaries of the main results of this paper.

2. RESULTS

In this section, \( m \) and \( n \) will be two fixed positive integers (unless specified otherwise). We start with the following.

Definition 2.1. A right \( R \)-module \( M \) is called \( (m, n) \)-injective if every right \( R \)-homomorphism from an \( n \)-generated submodule of \( R^m \) (or \( R_n \)) to \( M \) extends to one from \( R^m \) (or \( R_n \)) to \( M \). The ring \( R \) is a right \( (m, n) \)-injective ring if \( R \) is \( (m, n) \)-injective.

It is easy to see that \( M_R \) is \( (m, n) \)-injective if and only if \( M_R \) is \( (m, k) \)-injective for all \( 1 \leq k \leq n \) if and only if \( M_R \) is \( (l, n) \)-injective for all \( 1 \leq l \leq m \) if and only if \( M_R \) is \( (l, k) \)-injective for all \( 1 \leq l \leq m \) and \( 1 \leq k \leq n \).

A module \( M_R \) is called \( n \)-injective if every right \( R \)-homomorphism from an \( n \)-generated right ideal to \( M \) extends to one from \( R \) to \( M \), while \( M_R \) is \( f \)-injective [6] (\( = \)f.g.injective in [2] = Coflat in [5]) in case every right \( R \)-homomorphism from a finitely generated right ideal to \( M \) extends to one from \( R \) to \( M \). We call \( M_R \) a \( P \)-injective module if every right \( R \)-homomorphism from a right \( R \)-module \( aR \) to \( M \), \( a \in R \), extends to \( R \to M \). A module \( M_R \) is \( FP \)-injective [8] in case, for every finitely generated submodule \( K \) of a free right \( R \)-module \( F \), every homomorphism from \( K \) to \( M \) extends to one from \( F \) to \( M \). The ring \( R \) is right \( n \)-injective (resp. \( f \)-injective, \( P \)-injective, \( FP \)-injective) in case \( R \) is \( n \)-injective (resp. \( f \)-injective, \( P \)-injective, \( FP \)-injective).

The next lemma is immediate.

Lemma 2.2. Let \( M \) be a right \( R \)-module.

1. \( M \) is \( n \)-injective (resp. \( P \)-injective) if and only if \( M \) is \( (1, n) \)-injective (resp. \( (1, 1) \)-injective).
2. \( M \) is \( f \)-injective if and only if \( M \) is \( (1, n) \)-injective for all positive integers \( n \).
3. \( M \) is \( FP \)-injective if and only if \( M \) is \( (m, n) \)-injective for all positive integers \( m \) and \( n \) if and only if \( M \) is \( (n, n) \)-injective for all positive integers \( n \).

Remark 2.3. The \( (m, n) \)-injective modules lie between \( P \)-injective modules and \( FP \)-injective modules. A right \( (m, n) \)-injective ring need not be left \( (m, n) \)-injective as shown by [3, Example 2]. Rutter ([17, Example 1]) has an example of right \( (1, 1) \)-injective which is not right \( (1, 2) \)-injective.
Let \( M \) be a right \( R \)-module and \( z = (r_1, r_2, \ldots, r_n) \in R^n \). In what follows, we write \( Mz = \{xz \mid x \in M\} \), where \( xz = (xr_1, xr_2, \ldots, xr_n) \in M^n \).

**Theorem 2.4.** The following conditions are equivalent for a right \( R \)-module \( M 

1. \( M \) is \((m, n)\)-injective.
2. \( l_{M}r_{R} \{x_1, x_2, \ldots, x_m\} = Mz_1 + Mz_2 + \cdots + Mz_m \) for any \( m \)-element subset \( \{x_1, x_2, \ldots, x_m\} \) of \( R^n \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( x_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in R^n \), \( i = 1, 2, \ldots, m \). Suppose \( x = (x_1, x_2, \ldots, x_n) \in l_{M}r_{R} \{x_1, x_2, \ldots, x_m\} \). Take \( \beta_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in R^m \), \( i = 1, 2, \ldots, n \), and define \( g : \beta_1R + \beta_2R + \cdots + \beta_nR \to M \) such that

\[
g\left( \sum_{i=1}^{n} \beta_{it_i} \right) = \sum_{i=1}^{n} x_{it_i} \quad \text{for} \quad t_i \in R, \ i = 1, 2, \ldots, n.
\]

If \( \sum_{i=1}^{n} \beta_{it_i} = 0 \), then \( \sum_{i=1}^{n} a_{ij}t_i = 0 \), \( j = 1, 2, \ldots, m \). Let \( x = (t_1, t_2, \ldots, t_n) \in R^n \). Then \( x_jx^T = 0 \), \( j = 1, 2, \ldots, m \), and so \( x^T \in r_{R} \{x_1, x_2, \ldots, x_m\} \). Hence \( \sum_{i=1}^{n} x_{it_i} = 0 \). This shows that \( g \) is well-defined. Since \( M \) is \((m, n)\)-injective, \( g \) extends to a right \( R \)-homomorphism \( \tilde{g} : R^m \to M \). Let \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^m \) (with 1 in the \( i \)th position and 0’s in all other positions), \( y_i = \tilde{g}(e_i), i = 1, 2, \ldots, m \), and \( y = (y_1, y_2, \ldots, y_m) \in M^n \). Then, for any \( u = (u_1, u_2, \ldots, u_m) \in R^m \), \( \tilde{g}(u) = y_1u_1 + y_2u_2 + \cdots + y_mu_m = yu^T \). Thus \( x_i = g(\beta_i) = \tilde{g}(e_i) = y_{i1} = \sum_{j=1}^{m} y_ja_{ij}, i = 1, 2, \ldots, n \), and hence,

\[
x = (x_1, x_2, \ldots, x_n) = \left( \sum_{j=1}^{m} y_ja_{1j}, \sum_{j=1}^{m} y_ja_{2j}, \ldots, \sum_{j=1}^{m} y_ja_{nj} \right)
\]

So \( l_{M}r_{R} \{x_1, x_2, \ldots, x_m\} \subseteq Mz_1 + Mz_2 + \cdots + Mz_m \). The reverse inclusion is clear.

(2) \( \Rightarrow \) (1). Let \( N = \beta_1R + \beta_2R + \cdots + \beta_nR \) be an \( n \)-generated submodule of \( R^n \) and \( f : N \to M \) a right \( R \)-homomorphism. Write \( \beta_i = (a_{i1}, a_{i2}, \ldots, a_{im}) \in R^m \), \( i = 1, 2, \ldots, m \). Let \( u_i = f(\beta_i), i = 1, 2, \ldots, n \), and \( u = (u_1, u_2, \ldots, u_n) \). Then, for any \( \xi = (t_1, t_2, \ldots, t_n) \in r_{R} \{x_1, x_2, \ldots, x_m\} \), we have \( x_j\xi = 0 \), i.e., \( \sum_{i=1}^{n} a_{ij}t_i = 0, j = 1, 2, \ldots, m \). Thus \( \sum_{i=1}^{n} (a_{i1}, a_{i2}, \ldots, a_{im})t_i = 0 \), i.e., \( \sum_{i=1}^{n} \beta_{it_i} = 0 \), and so \( u_\xi = \sum_{i=1}^{n} u_{it_i} = \sum_{i=1}^{n} f(\beta_i)t_i = 0 \), whence \( u \in l_{M}r_{R} \{x_1, x_2, \ldots, x_m\} \). Therefore
by (2). Let \((u_1, u_2, \ldots, u_n) = (y_1x_1 + y_2x_2 + \cdots + y_mx_m)\) for some \(y_i \in M, i = 1, 2, \ldots, m\). Then

\[
(u_1, u_2, \ldots, u_n) = \left( \sum_{j=1}^{m} y_ja_{ij}, \sum_{j=1}^{m} y_ja_{2j}, \ldots, \sum_{j=1}^{m} y_ja_{nj} \right),
\]

and hence \(u_i = \sum_{j=1}^{m} y_ja_{ij} = y[i]^T, i = 1, 2, \ldots, n\), where \(y = (y_1, y_2, \ldots, y_m) \in M^m\). Now define \(f: R^m \rightarrow M\) such that \(f(x) = yx^T = \sum_{i=1}^{m} y_i x_i\) for each \(x = (x_1, x_2, \ldots, x_m) \in R^m\). Then \(f(\beta) = y[i]^T = u_i, i = 1, 2, \ldots, n\), and it follows that \(f\) is an extension of \(f\).

**Corollary 2.5.** The following statements hold for a module \(M_R\):

1. \(M_R\) is \(P\)-injective if and only if \(I_{M_R}(a) = Ma\) for all \(a \in R\).
2. \(M_R\) is \(n\)-injective if and only if \(I_{M_R}(a) = Mx\) for all \(a \in R^n\).
3. \(M_R\) is \(f\)-injective if and only if \(I_{M_R}(a) = Mx\) for all \(a \in R^n\) and for all positive integers \(n\).
4. \(M_R\) is \((m, 1)\)-injective if and only if \(I_{M_R}(I) = M1\) for every \(m\)-generated left ideal \(I\) of \(R\). In particular, \(R\) is right \((m, 1)\)-injective if and only if every \(m\)-generated left ideal of \(R\) is a left annihilator.

**Remark 2.6.** From Corollary 2.5 (4) we know that every finitely generated left ideal of \(R\) is a left annihilator if and only if \(R\) is right \((m, 1)\)-injective for all positive integers \(m\).

Recall that a ring \(R\) is left Kasch if every simple left \(R\)-module embeds in \(R\).

**Theorem 2.7.** Any left Kasch left \((n, m+1)\)-injective ring \(R\) is right \((m, n)\)-injective.

**Proof.** By Theorem 2.4, it is sufficient to prove that \(I_{R^n, R^m}(x_1, x_2, \ldots, x_m) = Rx_1 + Rx_2 + \cdots + Rx_m\) for all \(x_i \in R^n, i = 1, 2, \ldots, m\). Clearly, \(Rx_1 + Rx_2 + \cdots + Rx_m \subseteq I_{R^n, R^m}(x_1, x_2, \ldots, x_m)\). Suppose \(\beta \in I_{R^n, R^m}(x_1, x_2, \ldots, x_m)\), but \(\beta \notin I = Rx_1 + Rx_2 + \cdots + Rx_m\). Since \((R\beta + I)/I\) is a non-zero finitely generated left \(R\)-module, it has a maximal submodule \(M/I\). Hence \((R\beta + I)/M\) is a simple left \(R\)-module. Since \(R\) is left Kasch, let \(\delta: (R\beta + I)/M \rightarrow R\) be an embedding, and define \(f: R\beta + I \rightarrow R\) by \(f(x) = \delta(x + M)\) for \(x \in R\beta + I\). Clearly, \(f(I) = 0\) and \(f(\beta) \neq 0\). By hypothesis, \(f\) extends to a left \(R\)-homomorphism \(\bar{f}: R^n \rightarrow R\). Thus there exists
Corollary 2.8. The following statements hold for a ring R:

1. ([12, Theorem 3.1]). If R is left Kasch and left FP-injective, then R is right FP-injective.
2. ([14, Lemma 2.2]). If R is left Kasch and left 2-injective, then R is right P-injective.
3. ([2, Proposition 4.1]). Let R be left Kasch and left f-injective, then each finitely generated left ideal of R is a left annihilator.
4. If R is left Kasch and left (n, 2)-injective for all positive integers n, then R is right f-injective.

Theorem 2.9. The following conditions are equivalent for a module MR:

1. $M_R$ is $(m,n)$-injective.
2. $M_R$ is $(m,1)$-injective and $l_{M^n}(I \cap K) = l_{M^n}(I) + l_{M^n}(K)$, where I and K are submodules of $(R_m)_R$ such that $I + K$ is n-generated.
3. $M_R$ is $(m,1)$-injective and $l_{M^n}(I \cap K) = l_{M^n}(I) + l_{M^n}(K)$, where I and K are submodules of $(R_m)_R$ such that I is cyclic and K is $(n-1)$-generated (if $n=1, K=0$).

Proof. (1) $\Rightarrow$ (2). Clearly, $M_R$ is $(m,1)$-injective and

$$l_{M^n}(I) + l_{M^n}(K) \subseteq l_{M^n}(I \cap K).$$

Conversely, let $x \in l_{M^n}(I \cap K)$, then $f : I + K \to M$ is well defined by $f(c+b) = xc$ for all $c \in I$ and $b \in K$, so $f = y^\cdot$ for some $y = (y_1, y_2, \ldots, y_m) \in M^m$. Hence, for all $c \in I$ and $b \in K$, we have $yc = f(c) = xc$ and $yb = f(b) = 0$. Thus $x - y \in l_{M^n}(I)$ and $y \in l_{M^n}(K)$, so $x = (x - y) + y \in l_{M^n}(I) + l_{M^n}(K)$.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). We proceed by induction on n. Let $I = z_1R + z_2R + \cdots + z_nR$ be an n-generated submodule of $(R_m)_R$, $I_1 = z_1R$ and $I_2 = z_2R + \cdots + z_nR$. Suppose $f : I \to M$ is a right R-homomorphism. Then $f|_{I_1} = y_1^\cdot$ by hypothesis and $f|_{I_2} = y_2^\cdot$ by induction hypothesis for some $y_i \in M^m$, $i = 1, 2$. Thus $y_1 - y_2 \in l_{M^n}(I_1 \cap I_2) = l_{M^n}(I_1) + l_{M^n}(I_2)$, and so $y_1 - y_2 = z_1 + z_2$ for some $z_1 \in l_{M^n}(I_1)$, $i = 1, 2$. Let $y = y_1 - z_1 = y_2 + z_2$. Then $f = y^\cdot$. In fact, if $x \in I = I_1 + I_2$, then $x = z_1 + z_2$ with $z_i \in I_i$, $i = 1, 2,$
and so \( z_1x_1 = 0 \) and \( z_2x_2 = 0 \). Hence \( f(z) = f(x_1) + f(x_2) = y_1x_1 + y_2x_2 = (y_1 - z_1)x_1 + (y_2 + z_2)x_2 = y_1x_1 + y_2x_2 = y(z_1 + z_2) = yz \). So (1) follows. \( \square \)

**Corollary 2.10.** Let \( M \) be a right \( R \)-module.

1. The following conditions are equivalent:
   (a) \( M \) is \( n \)-injective.
   (b) \( M \) is \( P \)-injective and \( l_M(I \cap K) = l_M(I) + l_M(K) \), where \( I \) and \( K \) are right ideals of \( R \) such that \( I + K \) is \( n \)-generated.
   (c) \( M \) is \( P \)-injective and \( l_M(I \cap K) = l_M(I) + l_M(K) \), where \( I \) is a principal right ideal of \( R \) and \( K \) is an \((n-1)\)-generated right ideal of \( R \).
   In particular, \( M \) is \( 2 \)-injective if and only if \( M \) is \( P \)-injective and \( l_M(aR \cap bR) = l_M(a) + l_M(b) \) for all \( a, b \in R \).

2. ([6, Theorem 2.1]). \( M \) is \( f \)-injective if and only if \( M \) is \( P \)-injective and \( l_M(I \cap K) = l_M(I) + l_M(K) \) for each pair of finitely generated right ideals \( I \) and \( K \) of \( R \).

3. \( M \) is \((m, 2)\)-injective if and only if \( M \) is \((m, 1)\)-injective and
   \[ l_{M^m}(zR \cap \beta R) = l_{M^m}(z) + l_{M^m}(\beta) \]
   for \( z, \beta \in R_m \).

4. \( M \) is \( FP \)-injective if and only if \( l_Mr_R(I) = MI \) for all finitely generated left ideals \( I \) of \( R \) and \( l_{M^m}(H \cap K) = l_{M^m}(H) + l_{M^m}(K) \) for each pair of finitely generated submodules \( H \) and \( K \) of \((R_m)_R\) and for all positive integers \( m \).

In [8], Jain has shown that, if \( R \) is a right \( FP \)-injective ring, then every finitely generated left ideal is a left annihilator. This result can be improved as follows:

**Corollary 2.11.** A ring \( R \) is right \( FP \)-injective if and only if every finitely generated left ideal is a left annihilator and \( l_{R^m}(H \cap K) = l_{R^m}(H) + l_{R^m}(K) \) for each pair of finitely generated submodules \( H \) and \( K \) of \((R_m)_R\) and for all positive integers \( m \).

Recall that a ring \( R \) is called a left \( IN \)-ring [4] if \( r_R(H \cap K) = r_R(H) + r_R(K) \) for all left ideals \( H \) and \( K \) of \( R \). By [4, Example 16], an \( IN \)-ring need not be Kasch or \( P \)-injective. A ring \( R \) is called left simple-injective if every \( R \)-homomorphism with simple image from a left ideal of \( R \) to \( R \) is given by right multiplication by an element of \( R \). We also recall the following conditions:
Let injective and left continuous. If $R$ is a left $P$-injective and left IN-ring, then $R$ is left simple-continuous. The proof of the next Lemma is essentially due to Hajarnavis and Norton [7, Proposition 5.2].

**Lemma 2.12.** If $R$ is a left $P$-injective and left IN-ring, then $R$ is left simple-injective and left continuous.

**Proof.** Let $I$ be a left ideal of $R$ and $f: I \to R$ a homomorphism with simple image $f(I) = Ry$ for some $y \in R$. Choose $t \in I$ such that $f(t) = y$ and write $K = \text{Ker} f$. Then $I = Rt + K$. Since $R$ is left $P$-injective, $f_{Rt}: Rt \to R$ extends to $R$. Hence there exists $z \in R$ such that $f(x) = xz$ for all $x \in Rt$. Since $uz = f(u) = 0$ for all $u \in Rt \cap K$, $z \in r_R(Rt \cap K) = r_R(Rt) + r_R(K)$. Let $z = b + c$, where $b \in r_R(Rt)$ and $c \in r_R(K)$. For any $a \in I$, write $a = a_1 + a_2$, where $a_1 \in Rt$ and $a_2 \in K$. Then $a_1 b = 0 = a_2 c$, and so $f(a) = f(a_1) = a_1 z = a_1 c = ac$, i.e., $f = c$. Since $R$ is left $P$-injective, $R$ satisfies C2-condition by [14, Theorem 1.2]. On the other hand, $R$ is left quasi-continuous by [4, Theorem 5]. So $R$ is left continuous. 

**Theorem 2.13.** Let $R$ be a left Kasch, left $P$-injective and left IN-ring. Then every left ideal of $R$ is a left annihilator, and $R$ is right $f$-injective.

**Proof.** By Lemma 2.12 and [13, Lemma 4.2], every left ideal of $R$ is a left annihilator, and in particular, $R$ is right $P$-injective. By Corollary 2.10 (2), it is sufficient to prove that $I_R(H \cap K) = I_R(H) + I_R(K)$ for each pair of finitely generated right ideals $H$ and $K$ of $R$. In fact, since $R$ is a left $P$-injective and left IN-ring, $H = r_R I_R(H)$ and $K = r_R I_R(K)$ by [9, Lemma 5]. Clearly, $I_R(H) + I_R(K) \subseteq I_R(H \cap K)$. Suppose $I_R(H) + I_R(K) \neq I_R(H \cap K)$. Choose $b \in I_R(H \cap K)$ but $b \notin I = I_R(H) + I_R(K)$. Then $(Rb + L)/L$ has a maximal submodule $M/L$, and so $(Rb + L)/M$ is simple. Let $\sigma: (Rb + L)/M \to_R R$ be monic (for $R$ is left Kasch) and $f: Rb + L \to_R R$ be defined by $f(x) = \sigma(x + M)$ for $x \in Rb + L$. Then $\text{Im}(f)$ is simple. Thus $f = c$ for some $c \in R$. Since $R$ is left simple-injective by Lemma 2.12, and so $bc = f(b) \neq 0$. But $Mc = f(M) = 0$, and hence $Lc = 0$. Therefore $c \in r_R(L) = r_R(I_R(H) + I_R(K)) = r_R I_R(H) \cap r_R I_R(K) = H \cap K$, and so $bc = 0$, a contradiction.
Remark 2.14. We already know that a left $P$-injective and left $IN$-ring is left $f$-injective, and a left Kasch and left $FP$-injective ring is right $FP$-injective. But we wonder whether a left Kasch and left $f$-injective ring is right $f$-injective.

Theorem 2.15. The following conditions are equivalent for a right $R$-module $M$.

1. $M_R$ is $(m, n)$-injective.
2. If $z = (m_1, m_2, \ldots, m_n) \in M^n$ and $A \in R^{m \times n}$ satisfy $r_{R_n}(A) \subseteq r_{R_n}(z)$, then $z = yA$ for some $y \in M^m$.

Proof. (1) $\Rightarrow$ (2). Let $z = (m_1, m_2, \ldots, m_n) \in M^n$ and $A = (a_{ij}) \in R^{m \times n}$.

Put $a_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in R^n$, then $A = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix}$. Let $u \in r_{R_n}(a_1, a_2, \ldots, a_m)$.

Then $zu = 0$, $i = 1, 2, \ldots, m$, and hence $Au = 0$. Thus $u \in r_{R_n}(A) \subseteq r_{R_n}(z)$, and so $zu = 0$. It follows that

$$z \in I_Mr_{R_n}(a_1, a_2, \ldots, a_m) = Mz_1 + Mz_2 + \cdots + Mz_m$$

by Theorem 2.4. Therefore there exists $y_i \in M$, $i = 1, 2, \ldots, m$, such that

$$z = y_1z_1 + y_2z_2 + \cdots + y_mz_m = (y_1, y_2, \ldots, y_m)\begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} = yA,$$

where $y = (y_1, y_2, \ldots, y_m) \in M^m$.

(2) $\Rightarrow$ (1). Let $N = z_1R + z_2R + \cdots + z_nR$ be an $n$-generated submodule of $R^n_R$ and $f : N \to M$ a right $R$-homomorphism. Put $A = (z_1^T, z_2^T, \ldots, z_n^T) \in R^{m \times n}$, $m_i = f(z_i)$, $i = 1, 2, \ldots, n$, and $z = (m_1, m_2, \ldots, m_n) \in M^n$. Let $u = (u_1, u_2, \ldots, u_n)^T \in r_{R_n}(A)$. Then $Au = 0$, i.e., $z_1u_1 + z_2u_2 + \cdots + z_nu_n = 0$. Thus $x_1u_1 + x_2u_2 + \cdots + x_nu_n = 0$, and hence $zu = m_1u_1 + m_2u_2 + \cdots + m_nu_n = f(x_1u_1 + x_2u_2 + \cdots + x_nu_n) = 0$, i.e., $u \in r_{R_n}(z)$. By hypothesis, there exists $y = (y_1, y_2, \ldots, y_m) \in M^m$ such that $z = yA = y(z_1^T, z_2^T, \ldots, z_n^T)$, and then $m_i = yz_i^T$, $i = 1, 2, \ldots, n$. Define $f : R^m \to M$ such that $f(\xi^T) = y\xi^T$ for $\xi \in R^n$. Then $f(z_i) = yz_i^T = m_i = f(z_i)$, $i = 1, 2, \ldots, n$. So $f$ is an extension of $f$. \hfill $\square$

Corollary 2.16. The following statements hold:

1. The following conditions are equivalent:
   (a) $R$ is right $(n, n)$-injective.
   (b) If $z = (m_1, m_2, \ldots, m_n) \in R^n$ and $A \in R^{m \times n}$ satisfy $r_{R_n}(A) \subseteq r_{R_n}(z)$, then $z = yA$ for some $y \in R^n$. 

Proof. The equivalence \((a) \iff (b)\) follows from Theorem 2.15, and 
\((a) \iff (c)\) is by the remark following [15, Theorem 2.2]. (2) follows from (1) 
since the \((n,n)\)-injectivity of \(M_R\) implies the \(n\)-injectivity of \(M_R\).

\[\text{Proof.}\]

\(\text{Theorem 2.17. The following conditions are equivalent:}\)

\(1.\) \(R\) is right \((m, n)\)-injective.
\(2.\) \(R^m \rightarrow R^n \rightarrow R^N \rightarrow 0\) is exact, then \(N\) is torsionless.

\(\text{Proof.}\) (1) \(\Rightarrow\) (2). Let \(R^m \rightarrow R^n \rightarrow R^N \rightarrow 0\) be exact. Then there exists 
\(A \in M_{n \times m}(R)\) such that \(f(z) = zA\) for \(z \in R^m\), and so \(\text{Im}(f) = R^nA\), whence 
\(N \cong R^n/(R^nA)\). We will show that \(R^n/(R^nA)\) is torsionless. Let \(0 \neq \bar{z} \in R^n/(R^nA)\), where 
\(z = (z_1, z_2, \ldots, z_n) \in R^n \setminus (R^nA)\). By Theorem 2.15, 
\(r_{R^n}(A) \nsubseteq r_{R^n}(z)\). Thus there exists \(a = (a_1, a_2, \ldots, a_n)^T \in R_n\) such that 
\(Az = 0\) but \(\bar{z}z \neq 0\). Define \(g : R^n/(R^nA) \rightarrow R\) such that 
\(g(\bar{x}) = xz\) for every \(x \in R^n\). Clearly, \(g\) is well-defined, and \(g(\bar{z}) = \bar{z}z \neq 0\). So \(N \cong R^n/(R^nA)\) is 
torsionless.

(2) \(\Rightarrow\) (1). Let \(A \in R^{m \times n}\). Then \(N = R^n/(R^nA)\) is torsionless by (2) 
because \(N\) is the cokernel of \(f : R^m \rightarrow R^n\) defined by \(f(x) = xA\). Let 
\(z = (z_1, z_2, \ldots, z_n) \in R^n\). By Theorem 2.15, it is sufficient to show that, for 
\(z \notin R^nA\), \(r_{R^n}(A) \nsubseteq r_{R^n}(z)\). In fact, if \(z \notin R^nA\), then \(0 \neq \bar{z} \in R^n/(R^nA) = N\). 
Thus, there exists a left \(R\)-homomorphism \(g : R^n/(R^nA) \rightarrow R\) such that 
\(g(\bar{z}) \neq 0\) (for \(N\) is torsionless). Let \(e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^n\) (with 1 
in the \(i\)th position and 0’s in all other positions), \(i = 1, 2, \ldots, n\), and 
\(z = (g(e_1), g(e_2), \ldots, g(e_n)) \in R^n\). Then \(0 \neq \bar{z}z = g(\bar{z}) = g(e_1)\bar{z}_1 + z_2\bar{z}_2 + \cdots + 
z_n\bar{z}_n = z\alpha^T\), i.e., \(\bar{z} \notin r_{R^n}(z)\).

On the other hand, let \(e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^n\) (with 1 in the \(j\)th position and 0’s in all other positions), \(j = 1, 2, \ldots, m\). Note that \(g(\bar{x}) = xz^T\) 
for \(x \in R^n\). Thus \(e_jA\alpha^T = g(e_jA) = 0\) for \(j = 1, 2, \ldots, m\), and hence 
\(Az^T = 0\), i.e., \(z^T \in r_{R^n}(A)\). Therefore \(r_{R^n}(A) \nsubseteq r_{R^n}(z)\), as required. 

\[\text{Corollary 2.18. The following statements hold for a ring} R:\]

\(1.\) \(R\) is right \(n\)-injective if and only if the exactness of \(R^m \rightarrow R^n \rightarrow R^N \rightarrow 0\) implies the torsionlessness of \(N\).
\(2.\) The following conditions are equivalent:
\(a)\) \(R\) is right \(FP\)-injective.
\(b)\) Every finitely presented left \(R\)-module is torsionless.
\(c)\) For every positive integer \(n\), the exactness of \(R^n \rightarrow R^n \rightarrow R^n \rightarrow R^N \rightarrow 0\) implies the torsionlessness of \(N\).
Remark 2.19. The equivalence of (a) and (b) in Corollary 2.18 (2) is due to S. Jain [8, Theorem 2.3].

Recall that a ring $R$ is said to be right $(m, n)$-weakly linearly existentially closed (or $(m, n)$-wlec) [10] if every system of linear equations and a single linear inequation of the form

\[
\begin{align*}
    x_1a_1 + x_2a_2 + \cdots + x_na_n &= b_1 \\
    \vdots \\
    x_1a_1 + x_2a_2 + \cdots + x_na_n &= b_n \\
    x_1a_{n+1,1} + x_2a_{n+1,2} + \cdots + x_na_{n+1,n} &\neq b_{n+1}
\end{align*}
\]

which has a solution in some ring extension of $R$ has a solution in $R$ itself. A ring $R$ is right weakly linearly existentially closed (or wlec) if $R$ is right $(m, n)$-wlec for all positive integers $m$ and $n$. Left $(m, n)$-wlec rings and left wlec rings can be defined similarly.

Let $X = (x_1, x_2, \ldots, x_m)$, $A = (a_{ij}) \in R^{m \times n}$, $\gamma = (b_1, b_2, \ldots, b_n) \in R^n$ and $\alpha = (a_{n+1,1}, a_{n+1,2}, \ldots, a_{n+1,m})^T \in R_m$. The system above can be written in matrix form as

\[
\begin{align*}
    XA &= \gamma \\
    X\alpha &\neq b
\end{align*}
\]

where $b = b_{n+1} \in R$.

Theorem 2.20. The ring $R$ is right $(m, n)$-injective and left $(n, m)$-injective if and only if $R$ is right $(m, n)$-wlec and left $(n, m)$-wlec.

Proof. The proof is motivated by that of [10, Theorem 8].

"$\Rightarrow$". Let $A \in R^{m \times n}$, $X = (x_1, x_2, \ldots, x_m)$, $\alpha \in R_m$, $\gamma \in R^n$ and $b \in R$. If the system

\[
\begin{align*}
    XA &= \gamma \\
    X\alpha &\neq b
\end{align*}
\]

has a solution in the ring extension $S$ of $R$, i.e., there exists $X_0 \in S^m$ such that $X_0A = \gamma$ and $X_0\alpha \neq b$. Since $X_0A = \gamma$, $r_{R_S}(A) \subseteq r_{R_R}(\gamma)$. By Theorem 2.15, there exists $\delta_0 \in R^n$ such that $\gamma = \delta_0A$ (for $R_R$ is $(m, n)$-injective). We claim that there exists $\sigma_1 \in l_{R_S}(A)$ such that $(\delta_0 + \sigma_1)\beta \neq b$. Otherwise, $(\delta_0 + \sigma)\beta = b$ for all $\sigma \in l_{R_S}(A)$, and in particular, $\delta_0\beta = b$. It follows that $\sigma\beta = 0$ for all $\sigma \in l_{R_S}(A)$, and hence $l_{R_S}(A) \subseteq l_{R_S}(\beta)$. Therefore there exists
\( \beta \in R_n \) such that \( x = A \beta \) by Theorem 2.15 (for \( R \) is \((n, m)\)-injective), and so \( b = \delta_0 x = \delta_0 (A \beta) = (\delta_0 A) \beta = \gamma \beta = (X_0 A) \beta = X_0 (A \beta) = X_0 x \), a contradiction. Let \( \delta_1 = \delta_0 + \sigma_1 \). Then \( \delta_1 \in R^m \) and \( \delta_1 A = \delta_0 A = \gamma \) and \( \delta_1 x \neq b \), i.e., the system above has a solution in \( R \). So \( R \) is right \((m, n)\)-wlec. Similarly, \( R \) is left \((n, m)\)-wlec.

\( \text{"\( \Leftarrow \")}. \) We shall show that \( R_R \) is \((m, n)\)-injective. By Theorem 2.15, we have to show that if \( \beta \in R^n \) and \( A \in R^{m \times n} \) satisfy \( r_{R_R}(A) \subseteq r_{R_R}(\beta) \), then \( \beta = \xi A \) for some \( \xi \in R^n \).

First, let \( E \) be an \((R, R)\)-bimodule. Then we claim that \( r_{E_R}(A) \subseteq r_{E_R}(\beta) \). Let

\[
S = \left\{ \left( \begin{array}{cc} a & 0 \\ x & a \end{array} \right) \middle| a \in R, x \in E \right\}.
\]

We now consider the map

\[
a \to \tilde{a} = \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right)
\]

of \( R \) into \( S \). It is clear that this is a monomorphism of the ring \( R \) into \( S \). We shall identify \( R \) with its image in \( S \), identifying \( a \) with \( \tilde{a} \). In this way we can regard \( S \) as a ring extension of \( R \). Let \( A = (a_{ij}) \in R^{n \times n} \subseteq S^{m \times n} \) and \( \beta = (b_1, b_2, \ldots, b_n) \in R^n \subseteq S^n \). We write \( A = (\tilde{a}_{ij}) \in S^{m \times n} \) and \( \beta = (b_1, b_2, \ldots, b_n) \in S^n \). If \( r_{S_R}(A) \not\subseteq r_{S_R}(\beta) \), then there exists \( u \in S_n \) such that \( Au = 0 \) and \( \beta u \neq 0 \). Note that \( A \) (resp. \( \beta \)) is identified with \( A \) (resp. \( \beta \)). So the system

\[
\begin{align*}
AX &= 0 \\
\beta X &\neq 0
\end{align*}
\]

has a solution in \( S \). Since \( R \) is left \((n, m)\)-wlec, the above system has a solution in \( R \). Thus there exists \( v \in R_n \) such that

\[
\begin{align*}
Av &= 0 \\
\beta v &\neq 0,
\end{align*}
\]

which contradicts \( r_{R_R}(A) \subseteq r_{R_R}(\beta) \). So \( r_{S_R}(A) \subseteq r_{S_R}(\beta) \).

Now let \( u = (u_1, u_2, \ldots, u_n)^T \in r_{E_R}(A) \), then \( Au = 0 \). Put \( \bar{u}_i = \left( \begin{array}{c} 0 \\ u_i \\ 0 \end{array} \right) \in S, i = 1, 2, \ldots, n \), and \( \bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n)^T \). It follows that \( \bar{A} \bar{u} = 0 \). Thus \( \bar{u} \in r_{S_R}(A) \subseteq r_{S_R}(\beta) \), and so \( \beta \bar{u} = 0 \), whence \( \beta u = 0 \), i.e., \( u \in r_{E_R}(\beta) \). Therefore \( r_{E_R}(A) \subseteq r_{E_R}(\beta) \).
(m, n)-INJECTIVITY OF MODULES

Next let \( G \) be the \( \mathbb{Z} \)-injective envelope of the additive group of \( R \) and put \( E = \text{Hom}_{\mathbb{Z}}(R, G) \). It is easy to see that \( E \) is an \((R, R)\)-bimodule, \( E_R \) is injective and \( E \) is faithful. Let \( S = \text{End}_{R}(E_R) \), then \( E \) is a left \( S \)-module by defining \( s \cdot x = s(x) \) for \( s \in S \) and \( x \in E \). For any \( r \in R \), we define \( \tilde{r} \in S \) such that \( \tilde{r}(x) = r x \) for \( x \in E_R \). It is easy to see that the map \( r \mapsto \tilde{r} \) of \( R \) into \( S \) is a monomorphism. We shall now identify \( r \) with \( \tilde{r} \). Then \( R \) is identified with a subring of \( S \). By the first part of the proof, \( r_{E_{\gamma}}(A) \subseteq r_{E_{\gamma}}(\beta) \). Write \( AE_{\alpha} = \{ A_{\gamma} | \gamma \in E_{\alpha} \} \subseteq E_{\alpha} \) and define \( f : AE_{\alpha} \to E_R \) such that \( f(A_{\gamma}) = \beta_{\gamma} \), then \( f \) is a right \( R \)-homomorphism. Since \( E_R \) is injective, \( f \) extends to \( g : E_{m} \to E_{R} \). Let \( \lambda_{i} : E_{R} \to E_{m} \) be the \( i \)th injection and \( f_{i} = g \lambda_{i} \), then \( f_{i} \in S \), \( i = 1, 2, \ldots, m \). For any \( x = (a_{1}, a_{2}, \ldots, a_{n})^{T} \in E_{m} \), \( g(x) = g(\lambda_{1}(a_{1}) + \lambda_{2}(a_{2}) + \cdots + \lambda_{n}(a_{n})) = f_{1}(a_{1}) + f_{2}(a_{2}) + \cdots + f_{n}(a_{n}) = (f_{1}, f_{2}, \ldots, f_{n})x \).

Since \( g_{|AE_{\alpha}} = f_{\gamma} \) for any \( \gamma \in E_{\alpha} \), we have \( \beta_{\gamma} = f_{\gamma}(A_{\gamma}) = g_{\gamma} = (f_{1}, f_{2}, \ldots, f_{n})A_{\gamma} \). In particular, for any \( \gamma \in E_{\alpha} \), let \( \gamma_{i} = (0, \ldots, 0, x, 0, \ldots, 0)^{T} \in E_{\alpha} \) (with \( x \) in the \( i \)th position and 0's in all other positions), \( i = 1, 2, \ldots, n \). From \( (f_{1}, f_{2}, \ldots, f_{n})A_{\gamma_{i}} = \beta_{\gamma_{i}} \), we have \( \sum_{j=1}^{n} f_{j}(a_{j})x = b_{i}x \), i.e., \( \sum_{j=1}^{n} f_{j} \delta_{j}(x) = b_{i}(x) \) for all \( x \in E \), and so \( \sum_{j=1}^{n} f_{j} \delta_{j} = b_{i} \), \( i = 1, 2, \ldots, n \). Therefore \( (f_{1}, f_{2}, \ldots, f_{n})A = \beta \). Identifying \( A \) (resp. \( \beta \)) with \( A \) (resp. \( \beta \)), we have that the system \( YA = \beta \) has a solution in \( S \). Choose \( x \in R_{m} \) and \( b \in R \) such that

\[
(f_{1}, f_{2}, \ldots, f_{n})x \neq b.
\]

For example, take \( x = (1, 0, \ldots, 0)^{T} \), and

\[
b = \begin{cases} 
  1, & \text{if } f_{i} = 0 \\
  0, & \text{if } f_{i} \neq 0.
\end{cases}
\]

Thus the system

\[
\begin{align*}
YA &= \beta \\
Yx &\neq b
\end{align*}
\]

has a solution in \( S \), and hence it has a solution in \( R \) (for \( R \) is right \((m, n)\)-wlec). Therefore there exists \( \xi \in R^{n} \) such that \( \beta = \xi A \), as required. So \( R_{R} \) is \((m, n)\)-injective. Similarly, \( R_{R} \) is \((n, m)\)-injective.

\[ \square \]

**Corollary 2.21.** The following statements hold for a ring \( R \):

1. \( R \) is left and right \( P \)-injective if and only if \( R \) is left and right \((1, 1)\)-wlec.
2. \( R \) is right \( f \)-injective and every finitely generated right ideal of \( R \) is a right annihilator if and only if \( R \) is right \((1, n)\)-wlec and left \((n, 1)\)-wlec for all positive integers \( n \).

3. The following conditions are equivalent:
   (a) \( R \) is left and right \( FP \)-injective.
   (b) \( R \) is left and right \( wlec \).
   (c) \( R \) is left and right \((n, n)\)-wlec for all positive integers \( n \).

Remark 2.22. The equivalence of (a) and (b) in Corollary 2.21 (3) is due to P. Menal and P. Vamos [10, Theorem 8].

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