RELATIVE $FP$-PROJECTIVE MODULES

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Let $R$ be a ring and $M$ a right $R$-module. $M$ is called $n$-$FP$-projective if $\text{Ext}^1(R,M) = 0$ for any right $R$-module $N$ of $FP$-injective dimension $\leq n$, where $n$ is a nonnegative integer or $n = \infty$. $v_R(M)$ is defined as $\sup\{n : M$ is $n$-$FP$-projective$\}$ and $v_R(M) = -1$ if $\text{Ext}^1(R,M) \neq 0$ for some $FP$-injective right $R$-module $N$. The right $v$-dimension $v\dim(R)$ of $R$ is defined to be the least nonnegative integer $n$ such that $v_R(M) \geq n$ implies $v_R(M) = \infty$ for any right $R$-module $M$. If no such $n$ exists, set $v\dim(R) = \infty$. The aim of this paper is to investigate $n$-$FP$-projective modules and the $v$-dimension of rings.

Key Words: Cotorsion theory; $v$-Dimension; $FP$-injective dimension; $n$-$FP$-Projective module.

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1. NOTATION

In this section, we shall recall some known notions and definitions which we need in the later sections.

Throughout this paper, $R$ is an associative ring with identity and all modules are unitary $R$-modules. We write $M_R$ ($_RM$) to indicate a right (left) $R$-module. If $\cdots \to P_1 \to P_0 \to M \to 0$ is a projective resolution of an $R$-module $M$, let $K_0 = M$, $K_1 = \ker(P_0 \to M)$, $K_i = \ker(P_{i-1} \to P_{i-2})$ for $i \geq 2$. The $n$th kernel $K_n$ ($n \geq 0$) is called the $n$th syzygy of $M$. As usual, $wD(R)$ stands for the weak global dimension of $R$. $pd(M)$, $id(M)$ and $fd(M)$ denote the projective, injective and flat dimensions of $M$ respectively. $\text{Hom}(M,N)$ ($\text{Ext}^n(M,N)$) means $\text{Hom}_R(M,N)$ ($\text{Ext}^n_R(M,N)$) for an integer $n \geq 1$, and similarly $\text{Tor}_n(M,N)$ denotes $\text{Tor}_n^R(M,N)$ unless otherwise specified. For other concepts and notations, we refer the reader to Anderson and Fuller (1974), Enochs and Jenda (2000), Wisbauer (1991), and Xu (1996).

Let $R$ be a ring and $M$ a right $R$-module.
The $FP$-injective dimension of modules was introduced in Mao and Ding (2005) to measure how far away a module is from being $FP$-projective. In this paper, we approach $FP$-projective modules from another point of view and introduce the concepts of $n$-$FP$-projective modules and $v$-dimensions of modules and rings. A right $R$-module $M$ is called $n$-$FP$-projective if $\text{Ext}^1(M, N) = 0$ for all right $R$-modules $N$. Following Enochs and Jenda (2000), the $FP$-injective dimension of $M$, denoted by $FP\text{-id}(M)$, is defined to be the smallest integer $n \geq 0$ such that $\text{Ext}^{n+1}(M, N) = 0$ for every finitely presented right $R$-module $F$ (if no such $n$ exists, set $FP\text{-id}(M) = \infty$), and $r.FP$-$\dim(R)$ is defined as $\sup\{FP\text{-id}(M) : M \text{ is a right } R\text{-module}\}$.

In Mao and Ding (2005), the $FP$-projective dimension $fpd(M)$ of $M$ is defined to be the smallest integer $n \geq 0$ such that $\text{Ext}^{n+1}(M, N) = 0$ for any $FP$-injective right $R$-module $N$. If no such $n$ exists, set $fpd(M) = \infty$. The right $FP$-projective dimension $rfpD(R)$ of a ring $R$ is defined as $\sup\{fpd(M) : M \text{ is a finitely generated right } R\text{-module}\}$. $M$ is called $FP$-projective if $fpd(M) = 0$. Clearly, $fpd(M)$ measures how far away a right $R$-module $M$ is from being $FP$-projective. Enochs (1976) proved that a finitely generated right $R$-module $M$ is finitely presented if and only if $\text{Ext}^1(M, N) = 0$ for any $FP$-injective right $R$-module $N$, and so a finitely generated $FP$-projective right $R$-module is finitely presented. It follows that $rfpD(R)$ measures how far away a ring $R$ is from being right Noetherian.

A ring $R$ is called right coherent if every finitely generated right ideal of $R$ is finitely presented.

A pair $(\mathcal{F}, \mathcal{C})$ of classes of right $R$-modules is called a cotorsion theory (Enochs and Jenda, 2000) if $\mathcal{F}^\perp = \mathcal{C}$ and $\mathcal{C}^\perp = \mathcal{F}$, where $\mathcal{F}^\perp = \{C : \text{Ext}^1(F, C) = 0 \text{ for all } F \in \mathcal{F}\}$, and $\mathcal{C}^\perp = \{F : \text{Ext}^1(F, C) = 0 \text{ for all } C \in \mathcal{C}\}$.

Let $\mathcal{C}$ be a class of right $R$-modules and $M$ a right $R$-module. A homomorphism $\phi : M \to F$ with $F \in \mathcal{C}$ is called a $\mathcal{C}$-preenvelope of $M$ (Enochs and Jenda, 2000) if for any homomorphism $f : M \to F'$ with $F' \in \mathcal{C}$, there is a homomorphism $g : F \to F'$ such that $g\phi = f$. Moreover, if the only such $g$ are automorphisms of $F$ when $F = F$ and $f = \phi$, the $\mathcal{C}$-preenvelope $\phi$ is called a $\mathcal{C}$-envelope of $M$. A $\mathcal{C}$-envelope $\phi : M \to F$ is said to have the unique mapping property (Ding, 1996) if for any homomorphism $f : M \to F'$ with $F' \in \mathcal{C}$, there is a unique homomorphism $g : F \to F'$ such that $g\phi = f$. Following Enochs and Jenda (2000), Definition 7.1.6, a monomorphism $\alpha : M \to C$ with $C \in \mathcal{C}$ is said to be a special $\mathcal{C}$-preenvelope of $M$ if $\text{coker}(\alpha) \in \mathcal{C}^\perp$. Dually we have the definitions of a (special) $\mathcal{C}$-precover and a $\mathcal{C}$-cover (with the unique mapping property). Special $\mathcal{C}$-preenvelopes (respectively special $\mathcal{C}$-precovers) are obviously $\mathcal{C}$-preenvelopes (resp. $\mathcal{C}$-precovers).

2. INTRODUCTION

The $FP$-projective dimension of modules was introduced in Mao and Ding (2005) to measure how far away a module is from being $FP$-projective. In this paper, we approach $FP$-projective modules from another point of view and introduce the concepts of $n$-$FP$-projective modules and $v$-dimensions of modules and rings. A right $R$-module $M$ is called $n$-$FP$-projective if $\text{Ext}^1(M, N) = 0$ for all right $R$-modules $N$ of $FP$-injective dimension $\leq n$, where $n$ is a nonnegative integer or $n = \infty$. We define $v_\mathcal{R}(M) = \sup\{n : M \text{ is } n\text{-}FP\text{-projective}\}$ and $v_\mathcal{R}(M) = -1$ if $\text{Ext}^1(M, N) \neq 0$ for some $FP$-injective right $R$-module $N$. The right $v$-dimension $r \cdot v\dim(R)$ of a ring $R$ is defined to be the least nonnegative integer $n$ such that $v_\mathcal{R}(M) \geq n$ implies $v_\mathcal{R}(M) = \infty$ for any right $R$-module $M$. If no such $n$ exists, set $r \cdot v\dim(R) = \infty$. The purpose of this paper is to investigate these new notions.
Let $R$ be a right coherent ring and $n$ a fixed nonnegative integer.

In Section 3, we prove that $(\mathcal{FP}_n, \mathcal{FP})$ is a cotorsion theory, moreover, every right $R$-module has a special $\mathcal{FP}_n$-preenvelope, and every right $R$-module has a special $\mathcal{FP}_n$-precover, where $\mathcal{FP}_n$ denotes the class of all right modules of $FP$-injective dimension $\leq n$ (all $n$-$FP$-projective right $R$-modules) (see Theorem 3.8). Some characterizations of $n$-$FP$-projective right $R$-modules are given in Proposition 3.11.

Section 4 is devoted to rings whose every $n$-$FP$-projective module is projective. It is shown that $wD(R) \leq n$ if and only if every $n$-$FP$-projective right $R$-module is projective if and only if every $0$-$FP$-projective right $R$-module is of projective dimension $\leq n$ if and only if every $n$-$FP$-projective right $R$-module has an $\mathcal{FP}_n$-envelope with the unique mapping property (see Theorem 4.1). In particular, $R$ is a right semi-hereditary ring if and only if every $0$-$FP$-projective right $R$-module has a monic $\mathcal{FP}_0$-cover (see Corollary 4.2), and $r.v$-$\dim(R) = wD(R)$ if $wD(R) < \infty$ (see Corollary 4.3).

In Section 5, we characterize rings with the finite right $v$-dimension. It is shown that $r$-$v$-$\dim(R) \leq n$ if and only if every $n$-$FP$-projective right $R$-module is $(n+1)$-$FP$-projective if and only if every right $R$-module with finite $FP$-injective dimension has $FP$-injective dimension $\leq n$ (see Theorem 5.1).

Section 6 studies how are the rings satisfying that every module is $n$-$FP$-projective. It is proven that every right $R$-module is $n$-$FP$-projective if and only if every right $R$-module with $FP$-injective dimension $\leq n$ has an $\mathcal{FP}_n$-cover with the unique mapping property (see Theorem 6.1). We conclude this paper by proving that $rfpD(R) \leq 1$ and $\mathcal{FP}_0$ is closed under direct products if and only if every right $R$-module has an epic $\mathcal{FP}_0$-envelope (see Theorem 6.3).

3. DEFINITION AND GENERAL RESULTS

We start with the following definition.

**Definition 3.1.** Let $R$ be a ring and $n$ a nonnegative integer or $\infty$. A right $R$-module $M$ is called $n$-$FP$-projective provided that $\Ext^1(M, N) = 0$ for any right $R$-module $N$ with $FP$-$id(N) \leq n$.

For a right $R$-module $M$, let $v_R(M) = \sup \{n : M \text{ is } n$-$FP$-projective$\}$. We define $v_R(M) = -1$ if $\Ext^1(M, N) \neq 0$ for some $FP$-injective right $R$-module $N$.

The right $v$-dimension of a ring $R$, denoted by $r.v$-$\dim(R)$, is defined to be the least nonnegative integer $n$ such that $v_R(M) \geq n$ implies $v_R(M) = \infty$ for any right $R$-module $M$. If no such $n$ exists, set $r.v$-$\dim(R) = \infty$.

**Remark 3.2.** (1) $0$-$FP$-projective modules were called $FP$-projective modules in Mao and Ding (2005) and finitely covered modules in Trlifaj (2000). Clearly, finitely presented $R$-modules are always $0$-$FP$-projective, and projective modules are exactly $\infty$-$FP$-projective modules.

(2) It is clear that $v_R(M) \geq n$ if and only if $M$ is $n$-$FP$-projective for an integer $n \geq 0$, and $v_R(M) = \infty$ if and only if $M$ is $m$-$FP$-projective for any integer $m \geq 0$ if and only if $\Ext^1(M, N) = 0$ for all right $R$-modules $N$ with $FP$-$id(N) < \infty$.

(3) If $r.FP$-$\dim(R) \leq n$, then the class of all $n$-$FP$-projective right $R$-modules and the class of all projective right $R$-modules are the same. Therefore it is always true that $r.v$-$\dim(R) \leq r.FP$-$\dim(R)$.
(4) Let $R$ be an $n$-FC ring, i.e., $R$ is a two-sided coherent ring with $FP-id(qR) \leq n$ and $FP-id(R_R) \leq n$ for some nonnegative integer $n$ (see Ding and Chen, 1996). If $M$ is an $n$-FP-projective right (or left) $R$-module, then $v_R(M) = \infty$. Indeed, this follows from the fact that $FP-id(N) \leq n$ if and only if $FP-id(N) < \infty$ for any right (or left) $R$-module $N$ (see Ding and Chen, 1993, Proposition 3.16).

(5) If $R$ is right coherent and $FP-id(M) = m$, then $\text{Ext}^{m+k}(F, M) = 0$ for each finitely presented right $R$-module $F$ and each $k \geq 1$.

**Lemma 3.3.** Let $R$ be a right coherent ring. If $M$ is an $n$-FP-projective right $R$-module for some integer $n \geq 0$, then $\text{Ext}^j(M, N) = 0$ for any integer $j \geq 2$ and any right $R$-module $N$ with $FP-id(N) \leq n + 1$.

**Proof.** For every right $R$-module $N$ of $FP$-injective dimension $\leq n + 1$, there is a short exact sequence $0 \to N \to E \to L \to 0$ with $E$ injective and $FP-id(L) \leq n$. Therefore $\text{Ext}^j(M, N) \cong \text{Ext}^j(M, L) = 0$ and the result follows by induction. $\square$

**Remark 3.4.** Lemma 3.3 shows that, if $R$ is a right coherent ring and $M$ an $n$-FP-projective right $R$-module, then $\text{Ext}^j(M, N) = 0$ for any integer $j \geq 1$ and any right $R$-module $N$ with $FP-id(N) \leq n$.

**Proposition 3.5.** Let $R$ be a right coherent ring and $0 \to A \to B \to C \to 0$ an exact sequence of right $R$-modules.

1. If $v_R(C) \geq 0$, then $v_R(A) \geq \inf\{v_R(B), v_R(C) + 1\}$.
2. If $v_R(B) \geq \inf\{v_R(A), v_R(C)\}$.
3. If $B = A \oplus C$, then $v_R(A \oplus C) = \inf\{v_R(A), v_R(C)\}$.

**Proof.** The exact sequence $0 \to A \to B \to C \to 0$ gives rise to the exactness of the sequence $\text{Ext}^1(C, N) \to \text{Ext}^1(B, N) \to \text{Ext}^1(A, N) \to \text{Ext}^2(C, N)$ for any right $R$-module $N$. Now the result follows from Lemma 3.3 by a standard homological algebra argument. $\square$

**Corollary 3.6.** Let $R$ be a right coherent ring.

1. The $n$th syzygy $K_n$ of every finitely presented right $R$-module is $n$-FP-projective.
2. Every finitely generated submodule of any finitely generated 1-FP-projective right $R$-module is 1-FP-projective. In particular, each finitely generated right ideal of $R$ is 1-FP-projective.
3. For any right $R$-module homomorphism $\alpha: M \to N$ with $M$ and $N$ finitely generated 1-FP-projective, $\ker(\alpha)$ is 1-FP-projective. Furthermore, if $M$ is 2-FP-projective, then $\ker(\alpha)$ is 2-FP-projective.
4. The dual module $M^* = \text{Hom}(M, R)$ of any finitely presented left $R$-module $M$ is 2-FP-projective.

**Proof.** (1) Let $M$ be a finitely presented right $R$-module. There is an exact sequence

$$0 \to K_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0$$
with each \( P_i \) projective \((0 \leq i \leq n-1)\). Let \( K_i = \ker(P_0 \rightarrow M)\), \( K_i = \ker(P_{i-1} \rightarrow P_{i-2})\), \(2 \leq i \leq n\). Then \(0 \rightarrow K_1 \rightarrow P_0 \rightarrow M \rightarrow 0\) is exact. Since \( v_R(M) \geq 0\), \( v_R(P_0) = \infty\) (for \( P_0 \) is projective), we have \( v_R(K_1) \geq v_R(M) + 1 \geq 1\) by Proposition 3.5 (1). Thus (1) follows by induction.

(2) Let \( N \) be a finitely generated submodule of any finitely generated 1-FP-projective right \( R \)-module \( M \). Then we have a short exact sequence \( 0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0\). Note that \( M/N \) is finitely presented (for every finitely generated 1-FP-projective module is FP-projective, and hence finitely presented), so \( v_R(M/N) \geq 0\). Hence \( v_R(N) \geq 1\) by Proposition 3.5 (1), i.e., \( N \) is 1-FP-projective. The rest is clear.

(3) Note that \( \text{im}(x) \leq N\), and \( \text{im}(z) \) is finitely generated, so \( M/\ker(z) \cong \text{im}(z) \) is 1-FP-projective by (2). Consider the exact sequence \( 0 \rightarrow \ker(z) \rightarrow M \rightarrow M/\ker(z) \rightarrow 0\). Then (3) follows from Proposition 3.5 (1).

(4) Let \( M \) be a finitely presented left \( R \)-module. Then there exists an exact sequence \( F_1 \rightarrow F_0 \rightarrow M \rightarrow 0\) with \( F_1 \) and \( F_0 \) finitely generated free, which gives rise to the exactness of the sequence \( 0 \rightarrow M^* \rightarrow F_0^* \rightarrow F_1^*\). Therefore (4) holds by (3). \(\square\)

Recall that a right \( R \)-module \( M \) is called Gorenstein projective if there is an exact sequence

\[ \widetilde{P} = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots \]

of projective right \( R \)-modules such that \( M = \ker(P^0 \rightarrow P^1)\), and \( \text{Hom}(\widetilde{P}, Q) \) is exact for any projective right \( R \)-module \( Q\) (see Enochs and Jenda, 2000, Definition 10.2.1). A ring \( R \) is called a Gorenstein ring if \( R \) is a left and right noetherian ring, \( id_R(R) < \infty\) and \( id_R(R) < \infty\). Furthermore, if \( id_R(R) \leq n\) and \( id(R_R) \leq n\) for an integer \( n \geq 0\), then \( R \) is called an \( n \)-Gorenstein ring.

We observe that, if \( R \) is a Gorenstein ring, then each Gorenstein projective module is \( m \)-FP-projective for any integer \( m \) with \( 0 \leq m < \infty\), furthermore, if \( R \) is an \( n \)-Gorenstein ring, then a right (or left) \( R \)-module \( M \) is \( m \)-FP-projective \((n \leq m < \infty)\) if and only if \( M \) is Gorenstein projective by Enochs and Jenda (2000, Theorem 9.1.10 and Corollary 11.5.3).

**Remark 3.7.** Obviously, if \( M \) is a projective right \( R \)-module, then \( v_R(M) = \infty\). However, the converse is false in general because \( v_R(M) = \infty\) for every Gorenstein projective module \( M \) over a Gorenstein ring as shown by the preceding observation.

It is well known that (the class of Gorenstein projective \( R \)-modules, the class of \( R \)-modules of finite projective dimension) is a cotorsion theory over any Gorenstein ring \( R \) (see Enochs and Jenda, 2000, Remark 11.5.10). Denote by \( \mathcal{FI}_n \) (or \( \mathcal{FI}_n^* \)) the class of all right modules of FP-injective dimension \( \leq n \) (all \( n \)-FP-projective right \( R \)-modules). Then we have:

**Theorem 3.8.** Let \( R \) be a right coherent ring and \( n \geq 0\). Then \( (\mathcal{FI}_n, \mathcal{FI}_n^*) \) is a cotorsion theory. Moreover, every right \( R \)-module has a special \( \mathcal{FI}_n \)-preenvelope, and every right \( R \)-module has a special \( \mathcal{FI}_n \)-precover.
Let $M$ be a right $R$-module. $M$ admits an injective resolution

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^{n-1} \rightarrow E^n \rightarrow \ldots$$

Write $L^n = \text{im}(E^{n-1} \rightarrow E^n)$, $L^0 = M$. Then $M \in \mathcal{F}_n$ if and only if $L^n$ is FP-injective. Note that the latter is equivalent to $\text{Ext}^1(R/I, L^n) = 0$ for all finitely generated right ideals $I$ of $R$ by Stenström (1970, Lemma 3.1). This means that $\text{Ext}^{n+1}(R/I, M) = 0$ by dimension shifting. Denote by $K_I$ the $n$th syzygy module of the cyclic finitely presented right $R$-module $R/I$. Then $\text{Ext}^{n+1}(R/I, M) = 0$ if and only if $\text{Ext}^1(K_I, M) = 0$. Thus $\mathcal{F}_n = \bigoplus K_I$, where the sum is over all finitely generated right ideals $I$ in $R$, and so the result follows from Eklof and Trlifaj (2001, Theorem 10) and Enochs and Jenda (2000, Definition 7.1.5).

**Remark 3.9.** (1) Let $m$ and $n$ be nonnegative integers such that $m < n$. If $M$ is $n$-FP-projective, then $M$ is $m$-FP-projective. However, the converse is not true in general. In fact, take $R$ to be a right coherent ring with $wD(R) = r.FP$-$dim(R) = n$, for example, let $R = S[X_1, X_2, \ldots, X_n]$, the ring of polynomials in $n$ indeterminates over a von Neumann regular ring $S$ (see Glaz, 1989). Then the class of all right $R$-modules $\mathcal{F}_n \neq \mathcal{F}_m$, so there exists an $m$-FP-projective right $R$-module which is not $n$-FP-projective by Theorem 3.8.

(2) It is known that $\mathcal{F}_n$-envelopes may not exist in general (see Trlifaj, 2000, Theorem 4.9). However, if $\mathcal{F}_n$ is closed under direct limits, then every right $R$-module has an $\mathcal{F}_n$-envelope and every right $R$-module has an $\mathcal{F}_n$-cover by Theorem 3.8 and Enochs and Jenda (2000, Theorem 7.2.6).

**Corollary 3.10.** Let $R$ be a right coherent ring. Suppose a right $R$-module $M$ is the union of a continuous chain $(M_\lambda)_{\lambda \leq \check{\lambda}}$ of submodules. If $M_0 = 0$, $M_\lambda/M_\mu$ is projective relative to each epimorphism $A \rightarrow B$, where $A$ and $B$ are FP-injective, whenever $\mu + 1 < \lambda$, then $M$ is 1-FP-projective.

**Proof.** Let $X$ be a right $R$-module with $FP$-$id(X) \leq 1$. Consider an exact sequence $0 \rightarrow X \rightarrow E \rightarrow Y \rightarrow 0$ with $E$ injective. Then $Y$ and $E$ are FP-injective. Applying the functor $\text{Hom}(M_{\lambda+1}/M_\lambda, -)$ to the sequence, one gets $\text{Ext}^1(M_{\lambda+1}/M_\lambda, X) = 0$ whenever $\lambda + 1 < \check{\lambda}$. So the result follows from Theorem 3.8 and Enochs and Jenda (2000, Corollary 7.3.5).

We end this section with the following characterizations of $n$-FP-projective $R$-modules.

**Proposition 3.11.** Let $R$ be a right coherent ring with $FP$-$id(R) \leq n$ for an integer $n \geq 0$. Then the following are equivalent for a right $R$-module $M$:

1. $M$ is $n$-FP-projective;
2. $M$ is projective with respect to every exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with $A \in \mathcal{F}_n$;
3. For every exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ with $F \in \mathcal{F}_n$, $K \rightarrow F$ is an $\mathcal{F}_n$-preenvelope of $K$;
(4) \( M \) is a cokernel of an \( \mathcal{F}_n \)-preenvelope \( K \to F \) with \( F \) projective;

(5) There exists a right \( R \)-module exact sequence

\[
\tilde{E} = \cdots \to E_1 \to E_0 \to E^n \to E^1 \to \cdots
\]

with \( M = \ker(E^0 \to E^1) \), \( FP-id(E^i) \leq n \), \( FP-id(E_i) \leq n \) and \( E_i \) projective, \( i = 0, 1, 2, \ldots \), such that \( \text{Hom}(\tilde{E}, N) \) is exact for all right \( R \)-modules \( N \) with \( FP-id(N) \leq n \);

(6) There exists a projective resolution \( \tilde{E} = \cdots \to E_1 \to E_0 \to M \to 0 \) such that \( \text{Hom}(\tilde{E}, N) \) is exact for all right \( R \)-modules \( N \) with \( FP-id(N) \leq n \).

**Proof.** (1) \( \Rightarrow \) (2) is trivial.

(2) \( \Rightarrow \) (1) For every \( N \in \mathcal{F}_n \), consider a short exact sequence \( 0 \to N \to E \to L \to 0 \) with \( E \) injective.

(3) \( \Rightarrow \) (3) is clear.

(3) \( \Rightarrow \) (4) Let \( 0 \to K \to P \to M \to 0 \) be an exact sequence with \( P \) projective. Note that \( FP-id(P) \leq n \) since \( FP-id(R_R) \leq n \), thus \( K \to P \) is an \( \mathcal{F}_n \)-preenvelope.

(4) \( \Rightarrow \) (1) By (4), there is an exact sequence \( 0 \to K \to P \to M \to 0 \), where \( K \to P \) is an \( \mathcal{F}_n \)-preenvelope with \( P \) projective. Hence there is an exact sequence \( \text{Hom}(P, N) \to \text{Hom}(K, N) \to \text{Ext}^1(M, N) \to 0 \) for each \( N \in \mathcal{F}_n \). Note that \( \text{Hom}(P, N) \to \text{Hom}(K, N) \to 0 \) is exact by (4). Hence \( \text{Ext}^1(M, N) = 0 \), as desired.

(5) \( \Rightarrow \) (5) Let \( \cdots \to E_1 \to E_0 \to M \to 0 \) be a projective resolution of \( M \). By hypothesis, \( FP-id(E_i) \leq n \), \( i = 0, 1, 2, \ldots \). Let \( N \) be any right \( R \)-module with \( FP-id(N) \leq n \). Since \( M \) is \( n \)-FP-projective, \( \text{Ext}^1(M, N) = 0 \) for any integer \( j \geq 1 \) by Remark 3.4. Therefore the sequence

\[
0 \to \text{Hom}(M, N) \to \text{Hom}(E_0, N) \to \text{Hom}(E_1, N) \to \cdots
\]

is exact. On the other hand, we can construct an exact sequence

\[
0 \to M \to E^0 \to E^1 \to \cdots,
\]

where \( M \to E^0 \), \( \text{coker}(M \to E^0) \to E^1 \), \( \text{coker}(E^{n-1} \to E^n) \to E^{n+1} \) for \( n \geq 1 \) are \( \mathcal{F}_n \)-preenvelopes by Theorem 3.8. Thus we have the following exact sequence

\[
\cdots \to \text{Hom}(E^1, N) \to \text{Hom}(E^0, N) \to \text{Hom}(M, N) \to 0.
\]

Let

\[
\tilde{E} = \cdots \to E_1 \to E_0 \to E^n \to E^1 \to \cdots
\]

Then \( \text{Hom}(\tilde{E}, N) \) is exact.

(5) \( \Rightarrow \) (6) is obvious.
(6) ⇒ (1) It follows since

\[ \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow M \rightarrow 0 \]

is a projective resolution of \( M \) and \( \text{Hom}(E_0, N) \rightarrow \text{Hom}(E_1, N) \rightarrow \text{Hom}(E_2, N) \) is exact for all right \( R \)-modules \( N \) with \( \text{FP-id}(N) \leq n \).

\[ \square \]

4. RINGS WHOSE EVERY \( n \)-FP-PROJECTIVE MODULE IS PROJECTIVE

It is well–known that a ring \( R \) is von Neumann regular if and only if every right \( R \)-module is \( \text{FP} \)-injective if and only if every finitely presented right \( R \)-module is projective (flat). So \( R \) is von Neumann regular if and only if every \( 0 \)-FP-projective right \( R \)-module is projective (flat). In addition, if \( R \) is a right coherent ring, then \( R \) is von Neumann regular if and only if every \( 0 \)-FP-projective right \( R \)-module is \( \text{FP} \)-injective (see Mao and Ding, 2005, Corollary 4.3).

In what follows, let \( \sigma_M : M \rightarrow \mathcal{F}\mathcal{J}_n(M) (\varepsilon_M : \mathcal{F}\mathcal{P}_n(M) \rightarrow M) \) denote the \( \mathcal{F}\mathcal{J}_n \)-envelope (\( \mathcal{F}\mathcal{P}_n \)-cover) of a right \( R \)-module \( M \). Now we have:

**Theorem 4.1.** Let \( R \) be a right coherent ring and \( n \) a fixed nonnegative integer. Then the following are equivalent:

1. \( r \cdot \text{FP-dim}(R) \leq n \);
2. \( wD(R) \leq n \);
3. Every \( n \)-FP-projective right \( R \)-module is projective;
4. Every \( n \)-FP-projective right \( R \)-module is flat;
5. \( \text{pd}(M) \leq n \) for every \( 0 \)-FP-projective right \( R \)-module \( M \);
6. \( \text{FP-id}(M) \leq n \) for every \( n \)-FP-projective right \( R \)-module \( M \);
7. Every (\( n \)-FP-projective) right \( R \)-module has an \( \mathcal{F}\mathcal{J}_n \)-envelope with the unique mapping property.

Moreover if \( n \geq 1 \), then the above conditions are also equivalent to

8. \( \text{pd}(M) \leq 1 \) (\( \text{fd}(M) \leq 1 \)) for every (\( n-1 \))-FP-projective right \( R \)-module \( M \);
9. Every (\( (n-1) \)-FP-projective) right \( R \)-module \( M \) has a monic \( \mathcal{F}\mathcal{J}_{n-1} \)-cover \( \phi : F \rightarrow M \).

**Proof.** (1) ⇔ (2) ⇔ (5) hold by Stenström (1970, Theorem 3.3), (1) ⇒ (3) ⇒ (4) and (1) ⇒ (7) are trivial.

(3) ⇒ (1) Since \( r \cdot \text{FP-dim}(R) \leq n \) is equivalent to \( \text{FP-id}(M) \leq n \) for every right \( R \)-module \( M \), the result follows from Theorem 3.8.

(4) ⇒ (2) Let \( M \) be any finitely presented right \( R \)-module. By Corollary 3.6 (1), the \( n \)th syzygy of \( M \) is \( n \)-FP-projective, and so it is flat by (4). Thus \( \text{fd}(M) \leq n \), which implies that \( wD(R) \leq n \) by Enochs and Jenda (2000, Theorem 8.4.20).

(6) ⇒ (1) Let \( M \) be a right \( R \)-module. By Theorem 3.8, \( M \) has a special \( \mathcal{F}\mathcal{P}_n \)-precover, and hence there is a short exact sequence \( 0 \rightarrow K \rightarrow N \rightarrow M \rightarrow 0 \), where \( \text{FP-id}(K) \leq n \) and \( N \) is \( n \)-FP-projective. Since \( \text{FP-id}(N) \leq n \) by (6), \( \text{FP-id}(M) \leq n \). So (1) follows.
(7) \Rightarrow (6) Let \( M \) be an \( n \)-FP-projective right \( R \)-module. There is the following exact commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & M \\
\downarrow & & \sigma_M \\
0 & \rightarrow & \mathcal{F}_n(M) \\
& & \gamma \\
& & L \\
& & \rightarrow 0 \\
\downarrow & & \sigma_L \\
& & \mathcal{F}_n(L),
\end{array}
\]

where \( L \) is \( n \)-FP-projective by Wakamatsu’s Lemma in Xu (1996, Lemma 2.1.2). Note that \( \sigma_L \gamma \sigma_M = 0 = 0 \sigma_M \), so \( \sigma_L \gamma = 0 \) by (7). Therefore \( L = \text{im}(\gamma) \subseteq \ker(\sigma_L) = 0 \), and hence \( M \in \mathcal{F}_n \). Thus (6) follows.

(1) \Rightarrow (5) The proof has appeared in Mao and Ding (2005, Theorem 4.2), and here we include it for completeness. Let \( M \) be a 0-FP-projective right \( R \)-module. Then \( M \) admits a projective resolution

\[
\cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.
\]

Let \( N \) be any right \( R \)-module. Since \( FP-id(N) \leq n \), by Stenström (1970, Lemma 3.1), there is an exact sequence

\[
0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^{n-1} \rightarrow E^n \rightarrow 0,
\]

where \( E^0, E^1, \ldots, E^n \) are FP-injective. Therefore we form the following double complex

\[
\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
0 & \rightarrow \text{Hom}(M, E^n) & \rightarrow \text{Hom}(P_0, E^n) \\
& & \cdots \\
& & \rightarrow \text{Hom}(P_n, E^n) \\
& & \rightarrow \cdots
\end{array}
\]

Note that all rows are exact, except for the bottom row since \( M \) is 0-FP-projective and all \( E^i \) are FP-injective. Also note that all columns are exact except for the left column since all \( P_i \) are projective.

Using a spectral sequence argument, we know that the following two complexes

\[
0 \rightarrow \text{Hom}(P_0, N) \rightarrow \text{Hom}(P_1, N) \rightarrow \cdots \rightarrow \text{Hom}(P_n, N) \rightarrow \cdots
\]
and

\[ 0 \to \text{Hom}(M, E^0) \to \text{Hom}(M, E^1) \to \cdots \to \text{Hom}(M, E^n) \to 0 \]

have isomorphic homology groups. Thus \( \text{Ext}^{n+j}(M, N) = 0 \) for all integers \( j \geq 1 \), and hence \( p(M) \leq n \).

(1) \( \Rightarrow \) (8) Let \( M \) be an \((n-1)\)-FP-projective right \( R \)-module and \( N \) any right \( R \)-module. Since \( \text{FP}-\text{id}(N) \leq n \), \( \text{Ext}^2(M, N) = 0 \) by Lemma 3.3. Thus \( p(M) \leq 1 \).

(8) \( \Rightarrow \) (2) The proof is similar to that of (4) \( \Rightarrow \) (2).

(1) \( \Rightarrow \) (9) Let \( M \) be any right \( R \)-module. Write \( F = \sum \{ N \leq M : \text{FP}-\text{id}(N) \leq n - 1 \} \) and \( G = \bigoplus \{ N \leq M : \text{FP}-\text{id}(N) \leq n - 1 \} \). Then there exists an exact sequence 

\[ 0 \to K \to G \to F \to 0. \]

Since \( \text{FP}-\text{id}(K) \leq n \) by (1) and \( \text{FP}-\text{id}(G) \leq n - 1 \), we have \( \text{FP}-\text{id}(F) \leq n - 1 \). Next, we prove that the inclusion \( i : F \to M \) is an \( \mathcal{F}_n \)-cover of \( M \). Let \( \psi : F' \to F \) with \( F' \in \mathcal{F}_n \) be an arbitrary right \( R \)-homomorphism. Note that \( \psi(F') \leq F \) by the proof above. Define \( \zeta : F' \to F \) via \( \zeta(x) = \psi(x) \) for \( x \in F' \). Then \( i \zeta = \psi \), and so \( i : F \to M \) is an \( \mathcal{F}_n \)-precover of \( M \). In addition, it is clear that the identity map \( I_F \) of \( F \) is the only homomorphism \( g : F \to F \) such that \( ig = i \), and hence (9) follows.

(9) \( \Rightarrow \) (6) Let \( M \) be any \( n \)-FP-projective right \( R \)-module. We shall show that \( \text{FP}-\text{id}(M) \leq n \). Indeed, by Theorem 3.8, there exists an exact sequence 

\[ 0 \to M \to E \to L \to 0 \]

with \( \text{FP}-\text{id}(E) \leq n - 1 \) and \( L \in \mathcal{F}_n \). Since \( L \) has a monic \( \mathcal{F}_n \)-cover \( \phi : F \to L \), there is \( \alpha : M \to E \) such that the following exact diagram is commutative.

\[
\begin{array}{ccc}
0 & \longrightarrow & M \\
\downarrow & & \downarrow \\
F & \longrightarrow & E \\
\downarrow \alpha & & \downarrow \phi \\
0 & \longrightarrow & L \\
\end{array}
\]

Thus \( \phi \) is epic, and hence it is an isomorphism. Therefore \( \text{FP}-\text{id}(L) = \text{FP}-\text{id}(F) \leq n - 1 \), and so \( \text{FP}-\text{id}(M) \leq n \), as desired. \( \Box \)

It is well known that a right coherent ring \( R \) is right semi-hereditary if and only if \( wD(R) \leq 1 \).

By specializing Theorem 4.1 to the case \( n = 1 \), we have

**Corollary 4.2.** Let \( R \) be a right coherent ring. Then the following are equivalent:

1. \( R \) is right semi-hereditary;
2. Every 1-FP-projective right \( R \)-module is projective (flat);
3. Every 0-FP-projective right \( R \)-module is of projective (flat) dimension \( \leq 1 \);
4. Every 1-FP-projective right \( R \)-module is of FP-injective dimension \( \leq 1 \);
5. Every (0-FP-projective) right \( R \)-module has a monic \( \mathcal{F}_0 \)-cover.
Corollary 4.3. Let $R$ be a right coherent ring with $wD(R) < \infty$. Then $wD(R) = r.FP$-dim$(R) = r.\nu$-dim$(R)$.

Proof. Stenström (1970, Theorem 3.3) shows that $r.FP$-dim$(R) = wD(R)$. By Remark 3.2 (3), $r.\nu$-dim$(R) \leq r.FP$-dim$(R)$. Conversely, let $r.\nu$-dim$(R) = n < \infty$. For any $n$-FP-projective right $R$-module $M$, we have $v_n(M) = \infty$, and so $\text{Ext}^1(M, N) = 0$ for any right $R$-module $N$ with $\text{FP-id}(N) < \infty$, which implies that $M$ is projective since $r.FP$-dim$(R) < \infty$. Therefore $r.FP$-dim$(R) \leq n$ by Theorem 4.1 (3). This completes the proof. \hfill $\square$

5. RINGS WITH FINITE RIGHT $\nu$-DIMENSION

In this section we characterize rings with finite right $\nu$-dimension.

Theorem 5.1. Let $R$ be a right coherent ring and $n$ a fixed nonnegative integer. Then the following are equivalent:

1. $r.\nu$-dim$(R) \leq n$;
2. Every $n$-FP-projective right $R$-module is $(n + 1)$-FP-projective;
3. Every $n$th syzygy of any finitely presented right $R$-module is projective relative to each epimorphism $B \to C$, where $B$ is FP-projective and $\text{FP-id}(C) \leq n$;
4. Every right $R$-module $M$ with $\text{FP-id}(M) \leq n + 1$ has FP-projective dimension $\leq n$;
5. Every right $R$-module with finite FP-projective dimension has FP-projective dimension $\leq n$;
6. Every $n$th syzygy of any finitely presented right $R$-module is $(n + 1)$-FP-projective;
7. Every $n$th syzygy of any finitely presented right $R$-module is $m$-FP-projective for any integer $m \geq n + 1$;
8. $\text{fd}(M^+) \leq n$ for any right $R$-module $M$ with $\text{FP-id}(M) \leq n + 1$, where $M^+ = \text{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$;
9. For any pure submodule $N$ of every right $R$-module $M$ with $\text{FP-id}(M) \leq n + 1$, $\text{FP-id}(M/N) \leq n$.

Proof. (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (5) and (7) $\Rightarrow$ (6) are clear.

(2) $\Rightarrow$ (4) holds by Theorem 3.8.

(5) $\Rightarrow$ (1) Let $M$ be a right $R$-module with $v_n(M) \geq n$, i.e., $M$ is $n$-FP-projective. For any right $R$-module $N$ with $\text{FP-id}(N) < \infty$, we have $\text{Ext}^1(M, N) = 0$ since $\text{FP-id}(N) \leq n$ by (5). So $v_n(M) = \infty$, as desired.

(1) $\Rightarrow$ (7) follows from Corollary 3.6 (1).

(6) $\Rightarrow$ (9) Let $N$ be a pure submodule of a right $R$-module $M$ with $\text{FP-id}(M) \leq n + 1$. Then the pure exact sequence $0 \to N \to M \to M/N \to 0$ gives rise to the split exact sequence $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$. Therefore $(M/N)^+$ is a direct summand of $M^+$. By Fieldhouse (1972, Theorem 2.2), we have $\text{fd}(M^+) = \text{FP-id}(M) \leq n + 1$. So $\text{fd}((M/N)^+) \leq n + 1$, and hence $\text{FP-id}(M/N) \leq n + 1$. Let $K$ be any finitely presented right $R$-module and $K_n$ an $n$th syzygy of $K$. Then $\text{Ext}^1(K_n, M/N) = 0$ by (6), and so $\text{Ext}^{n+1}(K, M/N) = 0$, which implies that $\text{FP-id}(M/N) \leq n$. 

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(9) \implies (4) holds by letting \( N = 0 \).

(6) \iff (8) Let \( N \) be a finitely presented right \( R \)-module and \( N_n \) an \( n \)th syzygy of \( N \). By Rotman (1979, Theorem 9.51) and the remark following it, \( \text{Tor}^m(N, M) \cong (\text{Ext}^{m+1}(N, M))^+ \) for any right \( R \)-module \( M \). Since \( \text{Ext}^{n+1}(N, M) \cong \text{Ext}^1(N_n, M) \), the equivalence follows.

(3) \implies (4) Let \( M \) be a right \( R \)-module with \( \text{FP-id}(M) \leq n + 1 \). There exists an exact sequence \( 0 \to M \to E \to N \to 0 \) with \( E \) injective and \( \text{FP-id}(N) \leq n \). Suppose that \( K \) is a finitely presented right \( R \)-module and \( K_n \) an \( n \)th syzygy of \( K \). Then \( \mathbb{E} \text{xt}^1(K_n, M) = 0 \) by (3), and hence \( \mathbb{E} \text{xt}^{n+1}(K, M) = 0 \), which means \( \text{FP-id}(M) \leq n \).

(4) \implies (3) Let \( f : B \to C \) be an epimorphism and \( A = \ker(f) \), where \( B \) is \( \text{FP} \)-injective and \( \text{FP-id}(C) \leq n \). The exactness of \( 0 \to A \to B \to C \to 0 \) shows that \( \text{FP-id}(A) \leq n + 1 \), and so \( \text{FP-id}(A) \leq n \) by (4). Let \( N_n \) be an \( n \)th syzygy of a finitely presented right \( R \)-module \( N \). Then \( N_n \) is \( n \)-\( \text{FP} \)-projective by Corollary 3.6 (1), and so \( \mathbb{E} \text{xt}^1(N_n, A) = 0 \). Thus (3) follows. \( \square \)

Let \( n = 0 \) in Theorem 5.1. One gets

**Corollary 5.2.** Let \( R \) be a right coherent ring. Then the following are equivalent:

(1) \( r.v\dim(R) = 0 \);
(2) Every 0-\( \text{FP} \)-projective right \( R \)-module is 1-\( \text{FP} \)-projective;
(3) Every finitely presented right \( R \)-module is projective relative to each epimorphism \( B \to C \), where \( B \) and \( C \) are \( \text{FP} \)-injective;
(4) For any short exact sequence \( 0 \to A \to B \to C \to 0 \) of right \( R \)-modules, if \( B \) and \( C \) are \( \text{FP} \)-injective, then \( A \) is \( \text{FP} \)-injective;
(5) Every right \( R \)-module with finite \( \text{FP} \)-injective dimension is \( \text{FP} \)-injective;
(6) Every finitely presented right \( R \)-module is 1-\( \text{FP} \)-projective;
(7) Every finitely presented right \( R \)-module is \( m \)-\( \text{FP} \)-projective for any integer \( m \geq 1 \);
(8) \( M^+ \) is flat for any right \( R \)-module \( M \) with \( \text{FP-id}(M) \leq 1 \);
(9) For any pure submodule \( N \) of every right \( R \)-module \( M \) with \( \text{FP-id}(M) \leq 1 \), the quotient \( M/N \) is \( \text{FP} \)-injective.

**Remark 5.3.** (1) Recall that a ring \( R \) is said to be right \( IF \) if every injective right \( R \)-module is flat (see Colby, 1975). A right coherent and right \( IF \) ring \( R \) satisfies the equivalent conditions in Corollary 5.2. In fact, every finitely presented right \( R \)-module \( M \) is a submodule of a finitely generated free right \( R \)-module by Colby (1975, Theorem 1), so \( M \) is \( n \)-\( \text{FP} \)-projective for any \( n \geq 1 \) by Corollary 3.6 (1).

(2) By Remark 3.2 (3), \( r.v\dim(R) \leq r.\text{FP-dim}(R) \). The inequality may be strict. In fact, let \( R = \mathbb{Z}_4 \). Then \( R \) is a \( QF \) ring with \( wD(R) = \infty \) by Rotman (1979, Exercise 9.2 and Theorem 9.22), and so \( \text{FP-dim}(R) = \infty \) by Stenström (1970, Theorem 3.3). On the other hand, since the class of \( (\text{FP}) \)-injective modules over a \( QF \) ring coincides with the class of projective modules, \( r.v\dim(R) = 0 \) by Corollary 5.2 (4). This example also shows that Corollary 4.3 does not hold for a right coherent ring \( R \) with \( wD(R) = \infty \).
(3) Let $R$ be a commutative ring. The $\lambda$-dimension $\lambda_R(M)$ of an $R$-module $M$ and the $\lambda$-dimension $\lambda\dim(R)$ of the ring $R$ have been widely studied (see Couchot, 2003 and Vasconcelos, 1976). It is well known that $R$ is noetherian if and only if $\lambda\dim(R) = 0$, and $R$ is coherent if and only if $\lambda\dim(R) \leq 1$. However the $\lambda$-dimension is completely different from the $\nu$-dimension defined here. In fact, let $R = \mathbb{Z}$, the ring of integers. Then $\lambda\dim(R) = 0$. It is easy to see that $\nu\dim(R) \leq 1$. However, in the exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$, both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective, but $\mathbb{Z}$ is not injective, so $\nu\dim(R) \neq 0$ by Corollary 5.2 (4). Thus $\nu\dim(\mathbb{Z}) = 0$ while $\lambda\dim(\mathbb{Z}) = \infty$.

6. RINGS SATISFYING EVERY MODULE IS n-FP-PROJECTIVE

It is easy to see that a ring $R$ is right noetherian if and only if every right $R$-module is $0$-FP-projective if and only if $\nu_R(M) \geq 0$ for every right $R$-module $M$, and $R$ is semisimple artinian if and only if every right $R$-module is $\infty$-projective.

Next we shall give characterizations of those rings satisfying every right $R$-module is $n$-FP-projective for a fixed nonnegative integer $n$.

**Theorem 6.1.** Let $R$ be a right coherent ring and $n$ a fixed nonnegative integer. Then the following are equivalent:

1. Every right $R$-module is $n$-FP-projective;
2. Every finitely generated right $R$-module is $n$-FP-projective;
3. Every cyclic right $R$-module is $n$-FP-projective;
4. Every right $R$-module of FP-injective dimension $\leq n$ is $n$-FP-projective;
5. Every right $R$-module of FP-injective dimension $\leq n$ is injective;
6. $\text{Ext}^1(M, N) = 0$ for all right $R$-modules $M$ and $N$ with $\text{FP-id}(M) \leq n$ and $\text{FP-id}(N) \leq n$;
7. $\text{Ext}^i(M, N) = 0$ for all $i \geq 1$ and all right $R$-modules $M$ and $N$ with $\text{FP-id}(M) \leq n$ and $\text{FP-id}(N) \leq n$;
8. Every right $R$-module $M$ (with $\text{FP-id}(M) \leq n$) has an $\mathcal{F}_n$-cover with the unique mapping property.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (1) $\Rightarrow$ (8) and (1) $\Rightarrow$ (4) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) are obvious. (1) $\Leftrightarrow$ (5) follows from Theorem 3.8.

(8) $\Rightarrow$ (4) Let $M$ be any right $R$-module with $\text{FP-id}(M) \leq n$. There is the following exact commutative diagram

$$
\begin{array}{ccccccccc}
\mathcal{F}_n(K) & \xrightarrow{\varepsilon_K} & K & \xrightarrow{z} & \mathcal{F}_n(M) & \xrightarrow{\varepsilon_M} & M & \longrightarrow & 0 \\
0 & \downarrow & \zeta & \downarrow & \zeta & \downarrow & \zeta & \downarrow & 0 \\
0
\end{array}
$$
Let \( K \in \mathcal{F}_n \). Note that \( \epsilon_M \alpha \epsilon_K = 0 = \epsilon_M 0 \), so \( \alpha \epsilon_K = 0 \) by (8). Therefore \( K = \text{im}(\epsilon_K) \subseteq \ker(\alpha) = 0 \), and so \( M \) is \( n \)-FP-projective, as required.

(4) \( \Rightarrow \) (1) For any right \( R \)-module \( M \), by Theorem 3.8, there is a short exact sequence \( 0 \to M \to N \to L \to 0 \), where \( \text{FP-id}(N) \leq n \) and \( L \) is \( n \)-FP-projective. Since \( N \) is \( n \)-FP-projective by (4), \( M \) is \( n \)-FP-projective by Proposition 3.5 (1). Hence (1) follows.

(3) \( \Rightarrow \) (5) Let \( M \) be any right \( R \)-module with \( \text{FP-id}(M) \leq n \) and \( J \) any right ideal of \( R \). Then \( \text{Ext}^1(R/J, M) = 0 \) by (3). Thus \( M \) is injective, as desired. \( \square \)

**Remark 6.2.** By Theorem 6.1, if \( n \geq 1 \), then every right \( R \)-module is \( n \)-FP-projective if and only if every right \( R \)-module is \( 1 \)-FP-projective if and only if every right \( R \)-module with finite \( FP \)-injective dimension is injective if and only if \( v_r(M) = \infty \) for every right \( R \)-module \( M \). Thus, right noetherian rings can be classified into three mutually exclusive types: (a) semisimple artinian rings; (b) rings \( R \) such that \( wD(R) \neq 0 \) and every right \( R \)-module is \( 1 \)-FP-projective; (c) rings \( R \) for which there is a right \( R \)-module \( N \) with \( v_r(N) = 0 \).

Recall that the right \( FP \)-projective dimension \( rfpD(R) \) of a ring \( R \) is defined as \( \sup\{fpd(M) : M \) is a finitely generated right \( R \)-module\} (see Mao and Ding, 2005). We conclude this paper with the following result which is of independent interest.

**Theorem 6.3.** Let \( R \) be a right coherent ring. Then the following are equivalent:

1. \( rfpD(R) \leq 1 \) and \( \mathcal{FP}_0 \) is closed under direct products;
2. Every right \( R \)-module has an epic \( \mathcal{FP}_0 \)-envelope.

**Proof.** (1) \( \Rightarrow \) (2) Let \( M \) be any right \( R \)-module and \( \{M_i\}_{i \in I} \) the set of all the submodules of \( M \) with \( M_i = M_i \in \mathcal{FP}_0 \). The index set in the following statements is \( I \). Let \( \pi : M \to M_i \cap M_i \) be the natural map. It is clear that \( \alpha : M_i \cap M_i \to M_{M_i} \) defined by \( x \mapsto x + M_i \) induces a monomorphism \( \beta : M_i \cap M_i \to \prod_{M_i} M_{M_i} \). Note that \( \prod_{M_i} M_i \in \mathcal{FP}_0 \), and so \( M_i \cap M_i \in \mathcal{FP}_0 \) by Mao and Ding (2005, Proposition 3.7).

Now let \( N \in \mathcal{FP}_0 \) and \( \delta : M \to N \) be any homomorphism. Since \( M_i \text{ker}(\delta) \cong \text{im}(\delta) \leq N \in \mathcal{FP}_0 \), \( M_i / \text{ker}(\delta) \in \mathcal{FP}_0 \). Thus \( \cap M_i \leq \text{ker}(\delta) \), and so there is \( \xi : M_i \cap M_i \to N \) such that \( \xi \pi = \delta \). Thus \( \pi \) is an \( \mathcal{FP}_0 \)-envelope of \( M \). Since \( \pi \) is epic, \( \pi \) is an\( \mathcal{FP}_0 \)-envelope of \( M \).

(2) \( \Rightarrow \) (1) For any family \( \{M_i\}_{i \in I} \subseteq \mathcal{FP}_0 \), \( \prod_{M_i} M_i \) has an epic \( \mathcal{FP}_0 \)-envelope \( \pi : \prod_{M_i} M_i \to M \) by (2). Let \( \pi_i : \prod_{M_i} M_i \to M_i \) be the canonical projection. Then there is \( \beta_i : M_i \to M_i \) such that \( \beta_i \pi = \pi_i \). On the other hand, there is \( \gamma : M \to \prod_{M_i} M_i \) such that \( \pi_i \gamma = \beta_i \). Therefore \( \pi_i \gamma = \beta_i \pi = \pi_i \), and so \( \gamma \pi = I \prod_{M_i} M_i \). Thus \( \gamma \) is monic, and hence \( \prod_{M_i} M_i \cong M \in \mathcal{FP}_0 \).

Now let \( M \) be any right \( R \)-module. Then, by Theorem 3.8, there exists an exact sequence \( 0 \to K \overset{\varphi}{\longrightarrow} N \to M \to 0 \), where \( K \) is \( FP \)-injective and \( N \in \mathcal{FP}_0 \). By (2), \( K \) has an epic \( \mathcal{FP}_0 \)-envelope \( \gamma : K \to L \). Since \( N \in \mathcal{FP}_0 \), there is \( \theta : L \to N \)}
such that the following exact diagram is commutative

\[
\begin{array}{ccc}
0 & \longrightarrow & K \\
& \downarrow_{\gamma} & \downarrow_{\varphi} \\
L & \longrightarrow & N \\
& \downarrow & \downarrow \\
0 & \longrightarrow & M \\
& \downarrow & \downarrow \\
& 0 & \longrightarrow \\
\end{array}
\]

It follows that \( \gamma \) is monic since \( \varphi \) is monic. Therefore \( \gamma \) is an isomorphism, and so \( K \cong L \in \mathcal{FP}_0 \). Thus \( fpd(M) \leq 1 \) by Mao and Ding (2005, Proposition 3.1), and hence \( rfpD(R) \leq 1 \). This completes the proof. \( \square \)

**Remark 6.4.** The proof of Theorem 6.3 shows that, if \( R \) is a right coherent ring such that \( \mathcal{FP}_0 \) is closed under direct products, then \( rfpD(R) \leq 1 \) if and only if every \( FP \)-injective right \( R \)-module has an epic \( \mathcal{FP}_0 \)-envelope.

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