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RELATIVE PROJECTIVE MODULES AND RELATIVE INJECTIVE MODULES

Lixin Mao
Department of Basic Courses, Nanjing Institute of Technology, Nanjing, China and Department of Mathematics, Nanjing University, Nanjing, China

Nanqing Ding
Department of Mathematics, Nanjing University, Nanjing, China

Let R be a ring, and n and d fixed non-negative integers. An R-module M is called (n, d)-injective if $\mathrm{Ext}_R^{d+1}(P, M) = 0$ for any n-presented R-module P. M is said to be (n, d)-projective if $\mathrm{Ext}_R^1(M, N) = 0$ for any (n, d)-injective R-module N. We use these concepts to characterize n-coherent rings and (n, d)-rings. Some known results are extended.

Key Words: Cotorsion theory; n-coherent ring; (n, d)-injective module; (n, d)-projective module; (n, d)-ring.

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1. NOTATION

In this section, we shall recall some known notions and definitions that we will need in the later sections.

Throughout this article, R is an associative ring with identity and all modules are unitary R-modules. Let $\cdots \to P_1 \to P_0 \to M \to 0$ be a projective resolution of an R-module M, and $K_0 = M$, $K_1 = \ker(P_0 \to M)$, $K_i = \ker(P_{i-1} \to P_{i-2})$ for $i \geq 2$. The nth kernel K_n $(n \geq 0)$ is called the *nth syzygy of M*. Dually, we have the nth cosyzygy L^n of M using an injective resolution of M. $rD(R)$ and $wD(R)$ stand for the right global dimension and the weak global dimension of R, respectively. $pd(M)$, $fd(M)$, and $id(M)$ denote the projective, flat, and injective dimensions of M, respectively. $\mathrm{Hom}(M, N)$ (resp. $\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ (resp. $\mathrm{Ext}_R^n(M, N)$) for an integer $n \geq 1$.

Let R be a ring and n a non-negative integer. A right R-module M is called n-presented (see Chen and Ding, 1996a; Costa, 1994) if it has a finite n-presentation, i.e., there is an exact sequence of right R-modules

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0,$$

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Address correspondence to Nanqing Ding, Department of Mathematics, Nanjing University, Hankou Road No. 22, Nanjing 210093, China; E-mail: nqding@nju.edu.cn

where each F_i is finitely generated free (or projective). Clearly, every finitely generated projective R-module is n-presented for any $n \geq 0$. An R-module is 0-presented (resp. 1-presented) if and only if it is finitely generated (resp. finitely presented). Every m-presented R-module is n-presented for $m \geq n$. R is called *right n-coherent* (Costa, 1994) in case every n-presented right R-module is $(n+1)$-presented. It is easy to see that R is right 0-coherent (resp. 1-coherent) if and only if R is right Noetherian (resp. coherent), and every n-coherent ring is m-coherent for $m \geq n$. Following Costa (1994) and Zhou (2004), R is said to be a *right* (n, d)-*ring* if every n-presented right R-module has projective dimension at most d. (n, d)-rings stand for several known rings such as von Neumann regular, hereditary, semihereditary rings in case of different values of n, d. n-coherent rings and (n, d)-rings have been investigated by many authors (see Chen and Ding, 1996a; Costa, 1994; Dobbs et al., 1999; Mahdou, 2001; Zhou, 2004).

Let $\mathscr{C}$ be a class of right R-modules and M a right R-module. A homomorphism $\phi : M \to F$ with $F \in \mathscr{C}$ is called a *$\mathscr{C}$-preenvelope of M* (Enochs, 1981) if for any homomorphism $f: M \to F'$ with $F' \in \mathscr{C}$, there is a homomorphism $g : F \to F'$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of F when $F' = F$ and $f = \phi$, the $\mathscr{C}$-preenvelope ϕ is called a *$\mathscr{C}$-envelope of M*.

Given a class of right R-modules $\mathscr{L}$, we will denote by $\mathscr{L}^{\perp} = \{C : \mathrm{Ext}^1(L, C) = 0$ for all $L \in \mathscr{L}\}$ the right orthogonal class of $\mathscr{L}$, and by ${}^{\perp}\mathscr{L} = \{C : \mathrm{Ext}^1(C, L) = 0$ for all $L \in \mathscr{L}\}$ the left orthogonal class of $\mathscr{L}$. Following Enochs and Jenda (2000, Definition 7.1.6), a monomorphism $\alpha : M \to C$ with $C \in \mathscr{C}$ is said to be a special $\mathscr{C}$-preenvelope of M if $\mathrm{coker}(\alpha) \in {}^{\perp}\mathscr{C}$. Dually, we have the definitions of a (special) $\mathscr{C}$-precover and a $\mathscr{C}$-cover. Special $\mathscr{C}$-preenvelopes (resp. special $\mathscr{C}$-precovers) are obviously $\mathscr{C}$-preenvelopes (resp. $\mathscr{C}$-precovers). $\mathscr{C}$-envelopes ($\mathscr{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

A pair $(\mathscr{F}, \mathscr{C})$ of classes of right R-modules is called a *cotorsion theory* (Enochs and Jenda, 2000) if $\mathscr{F}^{\perp} = \mathscr{C}$ and ${}^{\perp}\mathscr{C} = \mathscr{F}$. A cotorsion theory $(\mathscr{F}, \mathscr{C})$ is called *complete* (Trlifaj, 2000) if every right R-module has a special $\mathscr{C}$-preenvelope, and every right R-module has a special $\mathscr{F}$-precover.

For further concepts and notations, we refer the reader to Anderson and Fuller (1974), Enochs and Jenda (2000), Rotman (1979), and Xu (1996).

2. INTRODUCTION

Let M be a right R-module. M is called *FP-injective* (Stenström, 1970) if $\mathrm{Ext}^1(N, M) = 0$ for all finitely presented right R-modules N. M is called *FP*-projective (Mao and Ding, 2005a) (or finitely covered Trlifaj, 2000) if $\mathrm{Ext}^1(M, N) = 0$ for any *FP*-injective right R-module N. It is well known that (the class of *FP*-projective right R-modules, the class of *FP*-injective right R-modules) is a complete cotorsion theory over any associative ring R (see Trlifaj, 2000, Theorem 3.4). The main purpose of this article is to extend the above-mentioned fact to a more general setting. Some applications are also given.

Let n and d be fixed non-negative integers. Recall that a right R-module M is called (n, d)-*injective* (Zhou, 2004) if $\mathrm{Ext}^{d+1}(P, M) = 0$ for any n-presented right R-module P. In Section 3, we introduce the concept of (n, d)-projective modules. A right R-module M is said to be (n, d)-*projective* if $\mathrm{Ext}^1(M, N) = 0$ for any (n, d)-injective right R-module N. After giving some basic properties of (n, d)-projective modules and (n, d)-injective modules, we prove that $(\mathscr{P}_{n,d}, \mathscr{I}_{n,d})$ is a complete cotorsion theory, where $\mathscr{I}_{n,d}$ (resp. $\mathscr{P}_{n,d}$) denotes the class of all (n, d)-injective (resp. (n, d)-projective) right modules (see Theorem 3.9).

Section 4 is devoted to some applications of our previous results. We first characterize n-coherent rings. It is shown that R is a right n-coherent ring if and only if every $(n, 0)$-projective right R-module is $(n+1, 0)$-projective if and only if $\mathscr{P}_{n,0}$ is closed under kernels of epimorphisms (see Theorem 4.1). Then we prove that a right n-coherent ring R is a right (n, d)-ring if and only if all $((n, d)$-projective) right R-modules are (n, d)-injective if and only if every (n, d)-projective right R-module is projective if and only if all $(n, 0)$-projective right R-modules are of projective dimension $\leq d$ if and only if $pd(M) \leq m$ for some m with $0 \leq m \leq n$ and any $(n-m, d)$-projective right R-module M (see Theorem 4.4). Dually, we give characterizations of those rings such that every right R-module is (n, d)-projective. It is shown that, for a right n-coherent ring R, every right R-module is (n, d)-projective if and only if every cyclic right R-module is (n, d)-projective if and only if every (n, d)-injective right R-module is (n, d)-projective if and only if every (n, d)-injective right R-module is injective (see Proposition 4.10).

3. DEFINITION AND GENERAL RESULTS

Let M be a right R-module, and n and d fixed non-negative integers. Recall that M is called *(n, d)-injective* (Zhou, 2004) if $\mathrm{Ext}^{d+1}(P, M) = 0$ for any n-presented right R-module P. Obviously, every (m, d)-injective right R-module is (n, d)-injective for every $m \leq n$; M is $(0, 0)$-injective (resp. $(1, 0)$-injective) if and only if M is injective (resp. FP-injective); M is $(0, d)$-injective if and only if $id(M) \leq d$; M is $(n, n-1)$-injective for some $n \geq 1$ if and only if M is n-FP-injective in sense of Chen and Ding (1996a).

Definition 3.1. Let n and d be non-negative integers. A right R-module M is said to be (n, d)-projective if $\mathrm{Ext}^1(M, N) = 0$ for any (n, d)-injective right R-module N.

In the following, we assume that n and d are non-negative integers. $\mathscr{P}_{n,d}$ (resp. $\mathscr{I}_{n,d}$) stands for the class of all (n, d)-projective (resp. (n, d)-injective) right modules.

Remark 3.2. (1) It is clear that any right R-module is $(0, 0)$-projective. Given a fixed integer d, every (n, d)-projective right R-module is (m, d)-projective for every $m < n$. A right R-module M is $(1, 0)$-projective if and only if M is FP-projective (Mao and Ding, 2005a). For a right coherent ring R, M is $(1, d)$-injective if and only if the FP-injective dimension of M is at most d (see Stenström, 1970, Lemma 3.1), hence a right R-module is $(1, d)$-projective if and only if it is d-FP-projective in sense of Mao and Ding (2005b).

(2) It is obvious that any n-presented right R-module M is $(n, 0)$-projective. Conversely, for $n \geq 1$, if M is finitely generated and $\mathrm{Ext}^{k+1}(M, N) = 0$ for any (1,0)-injective (i.e., FP-injective) right R-module N and any $0 \leq k \leq n-1$, then M is n-presented by Zhou (2004, Proposition 3.3).

Lemma 3.3. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence.*

(1) *If A is (n, d)-injective and B is $(n+1, d)$-injective, then C is $(n+1, d)$-injective.*
(2) *If B is (n, d)-projective and C is $(n+1, d)$-projective, then A is (n, d)-projective.*

Proof. (1) Let M be an $(n+1)$-presented right R-module. Then there is an exact sequence $0 \to K \to F \to M \to 0$ with F finitely generated free. Note that K is n-presented by Bourbaki (1985, p. 61, Exercise 6). Thus we get an induced exact sequence

$$0 = \mathrm{Ext}^{d+1}(K, A) \to \mathrm{Ext}^{d+2}(M, A) \to \mathrm{Ext}^{d+2}(F, A) = 0.$$

So $\mathrm{Ext}^{d+2}(M, A) = 0$. On the other hand, we have the exact sequence

$$0 = \mathrm{Ext}^{d+1}(M, B) \to \mathrm{Ext}^{d+1}(M, C) \to \mathrm{Ext}^{d+2}(M, A) = 0,$$

and hence $\mathrm{Ext}^{d+1}(M, C) = 0$. Therefore C is $(n+1, d)$-injective.

(2) Let N be any (n, d)-injective right R-module. Then we have the induced exact sequence

$$0 = \mathrm{Ext}^1(B, N) \to \mathrm{Ext}^1(A, N) \to \mathrm{Ext}^2(C, N).$$

Consider the short exact sequence $0 \to N \to E \to E/N \to 0$ with E injective. Note that E/N is $(n+1, d)$-injective by (1). Thus $\mathrm{Ext}^2(C, N) \cong \mathrm{Ext}^1(C, E/N) = 0$ since C is $(n+1, d)$-projective, and so $\mathrm{Ext}^1(A, N) = 0$, as required. □

Lemma 3.4. *The following are true for a right n-coherent ring R:*

(1) *For any $m \leq d$, every (n, m)-injective right R-module is (n, d)-injective, and every (n, d)-projective right R-module is (n, m)-projective*;
(2) *The mth cosyzygy of any $(n, m+d)$-injective right R-module is (n, d)-injective*;
(3) *If M is an (n, d)-projective right R-module, then $\mathrm{Ext}^{j+m+1}(M, N) = 0$ for any $j \geq 0$, $m \geq 0$ and any $(n, m+d)$-injective right R-module N*;
(4) *The mth syzygy of any (n, d)-projective right R-module is $(n, m+d)$-projective*;
(5) *A right R-module M is (n, d)-injective if and only if there exists an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^{d-1} \to E^d \to 0$, where each E^i is $(n,0)$-injective, $i = 0, 1, \ldots, d$*;
(6) *$\mathscr{I}_{n,d}$ is closed under cokernel of monomorphisms and $\mathscr{P}_{n,d}$ is closed under kernel of epimorphisms.*

Proof. (1) Let M be an (n, m)-injective right R-module and N an n-presented right R-module. Then the $(d-m)$th syzygy K_{d-m} of N is n-presented for any $m \leq d$ since R is a right n-coherent ring. Thus $\mathrm{Ext}^{d+1}(N, M) \cong \mathrm{Ext}^{m+1}(K_{d-m}, M) = 0$, and so M is (n, d)-injective. Therefore every (n, d)-projective right R-module is (n, m)-projective by definition.

(2) The proof is standard by dimension shifting.

(3) For every $(n, m+d)$-injective right R-module N, the mth cosyzygy L^m is (n, d)-injective by (2). Therefore $\mathrm{Ext}^{m+1}(M, N) \cong \mathrm{Ext}^1(M, L^m) = 0$ since M is (n, d)-projective. Consider the short exact sequence $0 \to N \to E \to L^1 \to 0$ with E injective. Since R is a right n-coherent ring, N is $(n, m+d+1)$-injective by (1), and so L^1 is $(n, m+d)$-injective by (2). Thus $\mathrm{Ext}^{m+2}(M, N) \cong \mathrm{Ext}^{m+1}(M, L^1) = 0$ by the preceding proof, and so the result follows by induction.

(4) Let M be any (n, d)-projective right R-module and K_m the mth syzygy of M. Note that $\mathrm{Ext}^1(K_m, N) \cong \mathrm{Ext}^{m+1}(M, N) = 0$ for any $(n, m+d)$-injective right R-module N by (3). Thus K_m is $(n, m+d)$-projective.

(5) The necessity follows from the fact that the dth cosyzygy of M is $(n, 0)$-injective by (2). The sufficiency holds by (1) and dimension shifting.

(6) It is easy to see that every $(n+1, d)$-injective right R-module is (n, d)-injective, and every (n, d)-projective right R-module is $(n+1, d)$-projective since R is a right n-coherent ring. So (6) follows from Lemma 3.3. □

Definition 3.5. A short exact sequence $0 \to A \to B \to C \to 0$ is called *n-pure* if $\mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to 0$ is exact for any n-presented module M. A submodule N of M is called an n-pure submodule if the sequence $0 \to N \to M \to M/N \to 0$ is n-pure.

Obviously, for an exact sequence $0 \to A \to B \to C \to 0$, it is 1-pure if and only if it is pure; it is 0-pure if and only if the epimorphism $B \to C$ is finitely split if and only if $\mathrm{im}(A \to B)$ is finitely split in B (see Azumaya, 1987, Theorem 3).

Proposition 3.6. *The following are equivalent for a right R-module M*:

(1) *M is $(n, 0)$-injective*;
(2) *Every exact sequence $0 \to M \to B \to C \to 0$ is n-pure*;
(3) *There exists an n-pure exact sequence $0 \to M \to B \to C \to 0$ with B $(n, 0)$-injective.*

If $n \geq 1$, then the above conditions are also equivalent to:

(4) *For any $(n-1)$-presented submodule N of any finitely generated free right R-module F, any homomorphism $f: N \to M$ extends to F.*

Proof. It is straightforward. □

Remark 3.7. Let $\{M_i\}_{i \in I}$ be a family of right R-modules with I an index set. It is clear that $\Pi_{i \in I} M_i$ is (n, d)-injective if and only if each M_i is (n, d)-injective. In addition, if $n \geq 1$, then $\bigoplus_{i \in I} M_i$ is $(n, 0)$-injective if and only if each M_i is $(n, 0)$-injective by Proposition 3.6 (4). Therefore, for a right n-coherent ring R, by Lemma 3.4 (5), $\bigoplus_{i \in I} M_i$ is (n, d)-injective if and only if each M_i is (n, d)-injective, which gives an alternative proof of Zhou (2004, Lemma 2.9).

Proposition 3.8. *The following are equivalent for a ring R and $n \geq 1$*:

(1) *For any n-pure submodule N of an injective right module M, the quotient M/N is injective*;
(2) *Every submodule of an* $((n, 0)$-$)$*projective right R-module is* $(n, 0)$*-projective*;
(3) *Every right ideal of R is* $(n, 0)$*-projective.*

Proof. (1) ⇒ (2) Let N be a submodule of an $(n, 0)$-projective right R-module M. Then, for any $(n, 0)$-injective right R-module L, we get an exact sequence

$$0 = \mathrm{Ext}^1(M, L) \to \mathrm{Ext}^1(N, L) \to \mathrm{Ext}^2(M/N, L).$$

Note that L is an n-pure submodule of its injective envelope $E(L)$ by Proposition 3.6, it follows that $E(L)/L$ is injective by (1) and $id(L) \leq 1$. Thus $\mathrm{Ext}^2(M/N, L) = 0$, and hence $\mathrm{Ext}^1(N, L) = 0$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let N be an n-pure submodule of an injective right module M. Then N is $(n, 0)$-injective by Proposition 3.6. For any right ideal I of R, the exactness of $0 \to N \to M \to M/N \to 0$ induces the exact sequence

$$0 = \mathrm{Ext}^1(R/I, M) \to \mathrm{Ext}^1(R/I, M/N) \to \mathrm{Ext}^2(R/I, N).$$

On the other hand, the exact sequence $0 \to I \to R \to R/I \to 0$ gives rise to the exact sequence

$$0 = \mathrm{Ext}^1(I, N) \to \mathrm{Ext}^2(R/I, N) \to \mathrm{Ext}^2(R, N) = 0$$

by (3). Thus $\mathrm{Ext}^2(R/I, N) = 0$, and hence $\mathrm{Ext}^1(R/I, M/N) = 0$, which implies that M/N is injective. $\square$

Theorem 3.9. *Let R be a ring. Then $(\mathscr{P}_{n,d}, \mathscr{I}_{n,d})$ is a complete cotorsion theory.*

Proof. Let M be any right R-module and K any n-presented right R-module. Note that $\mathrm{Ext}^{d+1}(K, M) = 0$ if and only if $\mathrm{Ext}^1(K_d, M) = 0$, where K_d denotes the dth syzygy of K. Let X be the set of representatives of dth syzygy modules of all n-presented right R-modules. Thus $\mathscr{I}_{n,d} = X^{\perp}$. Since $\mathscr{P}_{n,d} = {}^{\perp}(X^{\perp})$, the result follows from Eklof and Trlifaj (2001, Theorem 10) and Enochs and Jenda (2000, Definition 7.1.5). $\square$

Remark 3.10. (1) Note that $\mathscr{I}_{0,d}$ is the class of all right R-modules of injective dimension $\leq d$, and if R is right coherent, then $\mathscr{I}_{1,d}$ is the class of all right R-modules of FP-injective dimension $\leq d$. So Trlifaj (2000, Theorem 3.5) and Mao and Ding (2005b, Theorem 3.8) are immediate consequences of Theorem 3.9.

(2) Let R be a right n-coherent ring and $m < d$. By Lemma 3.4 (1), every (n, m)-injective right R-module is (n, d)-injective, and every (n, d)-projective right R-module is (n, m)-projective. However, the converse is not true in general. In fact, take R to be a right coherent ring with $wD(R) = d$, for example, let $R = S[X_1, X_2, \dots, X_d]$, the ring of polynomials in d indeterminates over a von Neumann regular ring S (see Glaz, 1989). Then, by Stenström (1970, Theorem 3.3), the class of all right R-modules $= \mathscr{I}_{1,d} \neq \mathscr{I}_{1,m}$, so there exists an $(1, m)$-projective right R-module which is not $(1, d)$-projective by Theorem 3.9.

(3) It is known that $\mathscr{I}_{0,0}$-envelopes always exist, but $\mathscr{I}_{1,0}$-envelopes may not exist in general (Trlifaj, 2000, Theorem 4.9). We note that if $\mathscr{P}_{n,d}$ is closed under direct limits, then every right R-module has an $\mathscr{I}_{n,d}$-envelope and a $\mathscr{P}_{n,d}$-cover by Theorem 3.9 and Enochs and Jenda (2000, Theorem 7.2.6).

4. APPLICATIONS

It is well known that a ring R is right 0-coherent (i.e., Noetherian) if and only if every right R-module is $(1, 0)$-projective if and only if every $(1, 0)$-injective right R-module is injective (see Mao and Ding, 2005a, Proposition 2.6). Similarly, we shall argue when $\mathcal{P}_{n,0}$ (resp. $\mathcal{I}_{n,0}$) coincides with $\mathcal{P}_{n+1,0}$ (resp. $\mathcal{I}_{n+1,0}$) for $n \geq 1$, which characterizes exactly right n-coherent rings as shown in the following theorem.

Theorem 4.1. *The following are equivalent for a ring R and $n \geq 1$:*

(1) *R is a right n-coherent ring;*
(2) *Every $(n+1, 0)$-injective right R-module is $(n, 0)$-injective;*
(3) *Every $(m, 0)$-injective right R-module with $m \geq 0$ is $(n, 0)$-injective;*
(4) *Every $(n, 0)$-projective right R-module is $(n+1, 0)$-projective;*
(5) *For a short exact sequence $0 \to A \to B \to C \to 0$, if A and B are $(n, 0)$-injective, then C is $(n, 0)$-injective;*
(6) *For a short exact sequence $0 \to A \to B \to C \to 0$, if B and C are $(n, 0)$-projective, then A is $(n, 0)$-projective;*
(7) $\mathrm{Ext}^k(M, \varinjlim N_i) \cong \varinjlim \mathrm{Ext}^k(M, N_i)$ *for any $k \geq 0$, any n-presented right R-module M and any direct system $\{N_i\}_{i \in I}$ of right R-modules with I directed;*
(8) *For any d, $\varinjlim N_i$ is (n, d)-injective for any direct system $\{N_i\}_{i \in I}$ of (n, d)-injective right R-modules with I directed.*

Proof. (1) $\Rightarrow$ (3) $\Rightarrow$ (2) are clear, and (2) $\Leftrightarrow$ (4) holds by Theorem 3.9.

(4) $\Rightarrow$ (6) follows from Lemma 3.3 (2).

(2) $\Rightarrow$ (5) By Lemma 3.3 (1), C is $(n+1, 0)$-injective since B is $(n+1, 0)$-injective. Thus C is $(n, 0)$-injective by (2).

(5) $\Rightarrow$ (1) Let M be any n-presented right R-module and N an $(1, 0)$-injective right R-module. Then there is an exact sequence of right R-modules

$$0 \to K_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where each F_i is finitely generated free, and K_n is finitely generated. Let L^n denote the nth cosyzygy of N. By (5), L^n is $(n, 0)$-injective since N is $(n, 0)$-injective. Then we have $\mathrm{Ext}^1(K_n, N) \cong \mathrm{Ext}^{n+1}(M, N) \cong \mathrm{Ext}^1(M, L^n) = 0$, and hence K_n is finitely presented by Enochs (1976) since K_n is finitely generated. Therefore M is $(n+1)$-presented, and so (1) holds.

(6) $\Rightarrow$ (5) For any $(n, 0)$-projective right R-module M, we have a short exact sequence $0 \to K \to P \to M \to 0$ with P projective. Thus K is $(n, 0)$-projective by (6), and we get an induced exact sequence

$$0 = \mathrm{Ext}^1(K, A) \to \mathrm{Ext}^2(M, A) \to \mathrm{Ext}^2(P, A) = 0.$$

Hence $\mathrm{Ext}^2(M, A) = 0$. On the other hand, the short exact sequence $0 \to A \to B \to C \to 0$ induces the exactness of the sequence

$$0 = \mathrm{Ext}^1(M, B) \to \mathrm{Ext}^1(M, C) \to \mathrm{Ext}^2(M, A) = 0.$$

Therefore $\mathrm{Ext}^1(M, C) = 0$, as desired.

(1) $\Rightarrow$ (7) Let M be any n-presented right R-module. Then M has a projective resolution $\cdots \to F_1 \to F_0 \to M \to 0$, where each F_i is finitely generated free since R is a right n-coherent ring. Therefore for any direct system $\{N_i\}_{i\in I}$ of right R-modules with I directed, we obtain a complex

$$0 \to \mathrm{Hom}(M, \varinjlim N_i) \to \mathrm{Hom}(F_0, \varinjlim N_i) \to \mathrm{Hom}(F_1, \varinjlim N_i) \to \cdots .$$

Note that $\mathrm{Hom}(F_k, \varinjlim N_i) \cong \varinjlim \mathrm{Hom}(F_k, N_i)$ for any $k \geq 0$. So by Enochs and Jenda (2000, Exercise 1.5.5, p. 35) or Rotman (1979, Exercise 6.4, p. 170), we have

$$\mathrm{Ext}^k(M, \varinjlim N_i) \cong \varinjlim \mathrm{Ext}^k(M, N_i)$$

for any $k \geq 0$, as desired.

(7) $\Rightarrow$ (8) is obvious.

(8) $\Rightarrow$ (1) follows from Chen and Ding (1996a, Theorem 3.1) by letting $d = n - 1$. The proof is complete. $\square$

Remark 4.2. (1) The equivalence of (1) through (3) in the above theorem has been proven by Zhou (see Zhou, 2004, Theorem 3.4). But it seems that there is a gap in the proof of (2) $\Rightarrow$ (1) in Zhou (2004, Theorem 3.4) because an n-presented right R-module need not be $(n+1)$-presented by Zhou (2004, Proposition 3.3).

(2) (1) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) has essentially been proven in Zhou (2004, Proposition 3.1). Here we prove the equivalence in a different way.

Let $n = 1$ in Theorem 4.1. One gets the following corollary.

Corollary 4.3. *The following are equivalent for a ring R:*

(1) *R is a right coherent ring*;
(2) *Every $(2, 0)$-injective right R-module is $(1, 0)$-injective*;
(3) *Every $(1, 0)$-projective right R-module is $(2, 0)$-projective*;
(4) *Every $(m, 0)$-injective right R-module with $m \geq 0$ is $(1, 0)$-injective*;
(5) *For a short exact sequence $0 \to A \to B \to C \to 0$, if A and B are $(1, 0)$-injective, then C is $(1, 0)$-injective*;
(6) *For a short exact sequence $0 \to A \to B \to C \to 0$, if B and C are $(1, 0)$-projective, then A is $(1, 0)$-projective*;
(7) *A right R-module N is $(1, 0)$-injective if and only if $\mathrm{Ext}^1(R/K, N) = 0$ for any finitely presented right ideal K of R.*

Proof. By Theorem 4.1, it is enough to show (1) $\Leftrightarrow$ (7).

(1) $\Rightarrow$ (7) The necessity is clear.

Conversely, let M be any finitely presented right R-module. We shall show that $\mathrm{Ext}^1(M, N) = 0$. We use induction on the number m of generators of M.

If $m = 1$, then $M \cong R/K$ with K finitely presented. So $\mathrm{Ext}^1(M, N) = 0$ by hypothesis.

Now suppose that the result is true for $m-1$, and $M = a_1R + a_2R + \cdots + a_mR$. Consider the exact sequence $0 \to a_1R \to M \to M/a_1R \to 0$, which induces the exactness of the sequence

$$\mathrm{Ext}^1(M/a_1R, N) \to \mathrm{Ext}^1(M, N) \to \mathrm{Ext}^1(a_1R, N) = 0.$$

Since $\mathrm{Ext}^1(M/a_1R, N) = 0$ by induction hypothesis, $\mathrm{Ext}^1(M, N) = 0$. Therefore N is $(1, 0)$-injective.

$(7) \Rightarrow (1)$ By Chen and Ding (1996a, Lemma 2.9), $\mathrm{Ext}^1(R/K, \varinjlim X_i) \cong \varinjlim \mathrm{Ext}^1(R/K, X_i)$ for any finitely presented right ideal K and any direct system $\{X_i\}_{i\in I}$ of $(1, 0)$-injective right R-modules with I directed. So $\varinjlim X_i$ is $(1, 0)$-injective by (7). Thus (1) follows from Stenström (1970, Theorem 3.2). □

In what follows, $\mathscr{P}_d$ (resp. $\mathscr{F}_d$) stands for the class of all right modules of projective (resp. flat) dimension $\leq d$. By Trlifaj (2000, Theorem 3.7) or Enochs and Jenda (2000, Theorem 7.4.6), $(\mathscr{P}_n, \mathscr{P}_n^{\perp})$ is a complete cotorsion theory. It is easy to verify that $M \in \mathscr{P}_n^{\perp}$ $(n \geq 1)$ if and only if M is injective with respect to every right R-module exact sequence $0 \to A \to B \to C \to 0$ with $A \in \mathscr{P}_{n-1}$ and B projective.

Theorem 4.4. *The following are equivalent for a right n-coherent ring R:*

(1) *R is a right (n, d)-ring;*
(2) *Every right R-module is (n, d)-injective;*
(3) *Every (n, d)-projective right R-module is (n, d)-injective;*
(4) *Every (n, d)-projective right R-module is projective;*
(5) *$M \in \mathscr{P}_d$ for every $(n, 0)$-projective right R-module M;*
(6) *Every right R-module M with $M \in \mathscr{P}_d^{\perp}$ is $(n, 0)$-injective;*
(7) *$M \in \mathscr{P}_m$ for any m with $0 \leq m \leq d$ and any $(n, d-m)$-projective right R-module M;*
(8) *$M \in \mathscr{P}_m$ for some m with $0 \leq m \leq d$ and any $(n, d-m)$-projective right R-module M;*
(9) *Every (n, d)-projective right R-module is flat;*
(10) *$M \in \mathscr{F}_d$ for every $(n, 0)$-projective right R-module M;*
(11) *$M \in \mathscr{F}_m$ for any m with $0 \leq m \leq d$ and any $(n, d-m)$-projective right R-module M;*
(12) *$M \in \mathscr{F}_m$ for some m with $0 \leq m \leq d$ and any $(n, d-m)$-projective right R-module M.*

Moreover if $n \geq 1$, $d \geq 1$, then the above conditions are also equivalent to:

(13) *Every $((n, d-1)$-projective$)$ right R-module M has a monic $\mathscr{I}_{n,d-1}$-cover.*

Proof. $(1) \Leftrightarrow (2)$, $(4) \Rightarrow (9)$, $(7) \Rightarrow (8)$, and $(7) \Rightarrow (11) \Rightarrow (12)$ are obvious.

$(2) \Leftrightarrow (4)$ is true by Theorem 3.9, $(9) \Rightarrow (10)$ holds by Lemma 3.4 (4).

$(5) \Leftrightarrow (6)$ follows from Theorem 3.9 and the fact that $(\mathscr{P}_k, \mathscr{P}_k^{\perp})$ is a cotorsion theory for any $k \geq 0$.

(3) $\Rightarrow$ (2) Let M be a right R-module. By Theorem 3.9, M has a special $\mathcal{P}_{n,d}$-precover, and hence there is a short exact sequence $0 \to K \to N \to M \to 0$, where $K \in \mathcal{I}_{n,d}$ and $N \in \mathcal{P}_{n,d}$. Since $N \in \mathcal{I}_{n,d}$ by (3), $M \in \mathcal{I}_{n,d}$ by Lemma 3.4 (6). So (2) follows.

(8) $\Rightarrow$ (5) Let M be an $(n, 0)$-projective right R-module. Then the $(d-m)$th syzygy K_{d-m} of M is $(n, d-m)$-projective by Lemma 3.4 (4). Therefore $\mathrm{Ext}^{d+1}(M, N) \cong \mathrm{Ext}^{m+1}(K_{d-m}, N) = 0$ for every right R-module N since $K_{d-m} \in \mathcal{P}_m$ by (8). So $M \in \mathcal{P}_d$.

(5) $\Rightarrow$ (2) Let M be any right R-module. It follows that $\mathrm{Ext}^{d+1}(F, M) = 0$ for any n-presented right R-module F since $pd(F) \leq d$, so M is (n, d)-injective.

(2) $\Rightarrow$ (7) Let M be an $(n, d-m)$-projective right R-module and N any right R-module. Since N is (n, d)-injective, $\mathrm{Ext}^{m+1}(M, N) = 0$ by Lemma 3.4 (3). Thus $pd(M) \leq m$.

(10) $\Rightarrow$ (1) Let M be an n-presented (and hence $(n, 0)$-projective) right R-module. Then the dth syzygy K_d of M is flat since $M \in \mathcal{F}_d$ by (10). But K_d is finitely presented since R is a right n-coherent ring. Thus K_d is projective.

(12) $\Rightarrow$ (10) The proof is similar to that of (8) $\Rightarrow$ (5).

(2) $\Rightarrow$ (13) Let M be any right R-module. Write $F = \sum\{N \leq M : N \text{ is } (n, d-1)\text{-injective}\}$ and $G = \bigoplus\{N \leq M : N \text{ is } (n, d-1)\text{-injective}\}$. Then there exists an exact sequence $0 \to K \to G \to F \to 0$. Since K is (n, d)-injective by (2) and G is $(n, d-1)$-injective by Remark 3.7, F is $(n, d-1)$-injective. Next we prove that the inclusion $i : F \to M$ is an $\mathcal{I}_{n,d-1}$-cover of M. Let $\psi : F' \to M$ with $F' \in \mathcal{I}_{n,d-1}$ be an arbitrary right R-homomorphism. Note that $\psi(F') \leq F$ by the proof above. Define $\zeta : F' \to F$ via $\zeta(x) = \psi(x)$ for $x \in F'$. Then $i\zeta = \psi$, and so $i : F \to M$ is an $\mathcal{I}_{n,d-1}$-precover of M. In addition, it is clear that the identity map 1_F of F is the only homomorphism $g : F \to F$ such that $ig = i$, and hence (13) follows.

(13) $\Rightarrow$ (3) Let M be any (n, d)-projective right R-module. We shall show that M is (n, d)-injective. Indeed, by Theorem 3.9, there exists an exact sequence $0 \to M \to E \xrightarrow{\beta} L \to 0$ with $E \in \mathcal{I}_{n,d-1}$ and $L \in \mathcal{P}_{n,d-1}$. Since L has a monic $\mathcal{I}_{n,d-1}$-cover $\phi : F \to L$ by (13), there is $\alpha : E \to F$ such that $\beta = \phi\alpha$. Thus ϕ is epic, and hence it is an isomorphism. Therefore L is $(n, d-1)$-injective, and so M is (n, d)-injective by Lemma 3.4 (3), as desired. □

By the proof of (2) $\Rightarrow$ (13) in Theorem 4.4, the condition $n \geq 1$ is used to guarantee that the direct sum of $(n, d-1)$-injective modules is $(n, d-1)$-injective. Note that R is a right noetherian ring if and only if any direct sum of injective right R-modules is injective if and only if every right R-module M has an injective (pre)cover (Enochs, 1981). So, by specializing Theorem 4.4 to the case $n = 0$, $d = 1$, we have the following (cf. Enochs and Jenda, 1991, Corollary 3.4).

Corollary 4.5. *The following are equivalent for a ring R:*

(1) *R is a right Noetherian right hereditary ring;*
(2) *R is a right Noetherian right $(0, 1)$-ring;*

(3) *R is right Noetherian and every* $(0, 1)$*-projective right R-module is projective*;
(4) *R is right Noetherian and every* $(0, 1)$*-projective right R-module is flat*;
(5) *R is right Noetherian and every* $(0, 1)$*-projective right R-module is* $(0, 1)$*-injective*;
(6) *Every right R-module M has a monic injective cover.*

By Corollary 4.4 and Stenström (1970, Theorem 3.3), we have the following theorem.

Corollary 4.6 (Mao and Ding, 2005b, Theorem 4.1). *The following are equivalent for a right coherent ring R*:

(1) *R is a right* $(1, d)$*-ring*;
(2) $wD(R) \leq d$;
(3) *Every* $(1, d)$*-projective right R-module is projective (flat)*;
(4) *Every* $(1, d)$*-projective right R-module is* $(1, d)$*-injective*;
(5) $pd(M) \leq d$ *for every* $(1, 0)$*-projective right R-module M.*

Moreover if $d \geq 1$, *then the above conditions are also equivalent to*:

(6) $pd(M) \leq 1$ $(fd(M) \leq 1)$ *for every* $(1, d-1)$*-projective right R-module M*;
(7) *Every* $((1, d-1)$*-projective) right R-module M has a monic* $\mathcal{I}_{1,d-1}$*-cover.*

Corollary 4.7. *The following are equivalent for a ring R*:

(1) *R is a right semihereditary ring*;
(2) *R is a right* $(1, 1)$*-ring*;
(3) *Every right R-module is* $(1, 1)$*-injective*;
(4) *Every* $(1, 1)$*-projective right R-module is projective*;
(5) $pd(M) \leq 1$ *for every* $(1, 0)$*-projective right R-module M*;
(6) *R is right coherent and every* $(1, 1)$*-projective right R-module is flat* $((1, 1)$*-injective)*;
(7) *R is right coherent and* $fd(M) \leq 1$ *for any* $(1, 0)$*-projective right R-module M*;
(8) *R is right coherent and every* $(1, 0)$*-projective right R-module M has a monic* $(1, 0)$*-injective cover*;
(9) *Every right R-module M has a monic* $(1, 0)$*-injective cover.*

Proof. (2) $\Leftrightarrow$ (3), and (5) $\Rightarrow$ (2) are clear.

(3) $\Leftrightarrow$ (4) follows from Theorem 3.9.

(1) $\Rightarrow$ (5) and (1) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) hold by Corollary 4.6.

(2) $\Rightarrow$ (1) is easy.

(1) $\Leftrightarrow$ (9) follows from Chen and Ding (1996b, Corollary 8). □

It is well known that a ring R is von Neumann regular if and only if every right R-module is $(1, 0)$-injective if and only if every finitely presented right R-module is projective (flat). So R is von Neumann regular if and only if every $(1, 0)$-projective right R-module is projective (flat). In addition, if R is a right coherent ring, then R is von Neumann regular if and only if every $(1, 0)$-projective right R-module is $(1, 0)$-injective (see Mao and Ding, 2005a, Corollary 4.3). Next we shall give characterizations of $(n, 0)$-rings (i.e., n-von Neumann regular rings in sense of Mahdou, 2001).

Bass proved that the dual module $M^* = \mathrm{Hom}(M, R) \neq 0$ for any nonzero finitely presented left R-module M if and only if every finitely generated projective submodule of a (finitely generated) projective right R-module is a direct summand (see Bass, 1960, Theorem 5.4).

Theorem 4.8. *The following are equivalent for a ring R and $n \geq 1$:*

(1) *R is a right $(n, 0)$-ring;*
(2) *Every right R-module is $(n, 0)$-injective;*
(3) *Every $(n, 0)$-projective right R-module is projective;*
(4) *Every $(n,0)$-projective right R-module is flat;*
(5) *R is a right n-coherent ring and every $(n,0)$-projective right R-module is $(n, 0)$-injective;*
(6) *Every short exact sequence $0 \to A \to B \to C \to 0$ is n-pure;*
(7) *Every $(n-1)$-presented submodule N of any finitely generated free right R-module is a direct summand;*
(8) *R is a right (n, d)-ring for some d with $1 \leq d \leq n$, and the dual module $M^* =$ $\mathrm{Hom}(M, R) \neq 0$ for any nonzero finitely presented left R-module M;*
(9) $\mathrm{Hom}(M, \varinjlim X_i) \cong \varinjlim \mathrm{Hom}(M, X_i)$ *for any n-presented right R-module M and any direct system $\{X_i\}_{i \in I}$ of right R-modules with I quasi-ordered (not necessarily directed).*

Moreover, if $rD(R) < \infty$, then the above conditions are also equivalent to:

(10) *R is a right n-coherent ring and R is $(n, 0)$-injective as a right R-module.*

Proof. (1) $\Leftrightarrow$ (2), and (3) $\Rightarrow$ (4) are clear.

(2) $\Leftrightarrow$ (3) follows from Theorem 3.9.

(4) $\Rightarrow$ (1) By (4), every n-presented right R-module M is flat, and hence projective (for M is finitely presented), as desired.

(2) $\Rightarrow$ (5) follows from Theorem 4.1 and (5) $\Rightarrow$ (2) holds by Theorem 4.4.

(2) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) hold by Proposition 3.6.

(1) $\Rightarrow$ (8) R is clearly a right (n, d)-ring for any d. Let A be a finitely generated projective submodule of a finitely generated projective right R-module B. Then B/A is projective since B/A is n-presented. So A is a direct summand of B, and hence (8) follows from Bass (1960, Theorem 5.4).

(8) $\Rightarrow$ (1) Let M be an n-presented right R-module. Then $M \in \mathscr{P}_d$ since R is a right (n, d)-ring for some d with $1 \leq d \leq n$, i.e., there is an exact sequence

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where each P_i is finitely generated projective by Bourbaki (1985, Exercise 6, p.61). By (8) and Bass (1960, Theorem 5.4), the exact sequence $0 \to P_d \to P_{d-1} \to K_{d-1} \to 0$

is split, and so K_{d-1} is finitely generated projective. Similarly, the syzygy modules $K_{d-2}, K_{d-3}, \ldots, K_1$ and M are finitely generated projective. Thus R is a right $(n, 0)$-ring.

(1) $\Rightarrow$ (9) Let M be an n-presented right R-module. Then M is projective by (1). Thus the functor $\mathrm{Hom}(M, -)$ is right exact. Note that $\mathrm{Hom}(M, -)$ preserves direct sums by Anderson and Fuller (1974, Exercise 16.3, p. 189) since M is finitely generated. Therefore $\mathrm{Hom}(M, -)$ preserves all direct limits (with any, not necessarily directed, quasi-ordered index set) by Rotman (1979, Theorem 3.35).

(9) $\Rightarrow$ (1) For any n-presented right R-module M, the functor $\mathrm{Hom}(M, -)$ is right exact by Rotman (1979, Theorem 3.35). Thus M is projective, and so (1) follows.

(2) $\Rightarrow$ (10) is obvious.

(10) $\Rightarrow$ (2) For any right R-module M, $pd(M) = m < \infty$ since $rD(R) < \infty$. Thus there exists an exact sequence

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where each P_i is projective, $i = 0, 1, \ldots, m$. Note that each P_i is $(n, 0)$-injective by Remark 3.7 since R is $(n, 0)$-injective as a right R-module. It follows that the syzygy modules $K_{m-1}, K_{m-2}, \ldots, K_1$ and M are $(n, 0)$-injective by Theorem 4.1. $\square$

A $\mathscr{C}$-cover $\phi : F \to M$ is said to have the unique mapping property (Ding, 1996) if for any homomorphism f: $F' \to M$ with $F' \in \mathscr{C}$, there is a unique homomorphism $g : F' \to F$ such that $\phi g = f$.

Proposition 4.9. *The following are equivalent for a right n-coherent ring R:*

(1) *Every right R-module is $(n, 2)$-injective, and every (resp. $(n, 2)$-projective) right R-module has an $(n, 0)$-injective cover;*
(2) *Every (resp. $(n, 2)$-projective) right R-module has an $(n, 0)$-injective cover with the unique mapping property.*

Proof. (1) $\Rightarrow$ (2) Let M be any (resp. $(n, 2)$-projective) right R-module. Then M has an $(n, 0)$-injective cover $f : F \to M$ by (1). It is enough to show that, for any $(n, 0)$-injective right R-module G and any homomorphism $g : G \to F$ such that $fg = 0$, we have $g = 0$. In fact, there exists $\beta : F/\mathrm{im}(g) \to M$ such that $\beta\pi = f$ since $\mathrm{im}(g) \subseteq \ker(f)$, where $\pi : F \to F/\mathrm{im}(g)$ is the natural map. Note that $F/\mathrm{im}(g)$ is $(n, 0)$-injective since $\ker(g)$ is $(n, 2)$-injective. Thus there exists $\alpha : F/\mathrm{im}(g) \to F$ such that $\beta = f\alpha$, and we get the following exact commutative diagram:

$$\begin{tikzcd}
 & & & M & & \\
0 \arrow[r] & \ker(g) \arrow[r, "i"] & G \arrow[r, "g"] \arrow[ur, "0"] & F \arrow[u, "f"] \arrow[r, shift left, "\pi"] & F/\mathrm{im}(g) \arrow[ul, "\beta"'] \arrow[l, shift left, "\alpha"] \arrow[r] & 0
\end{tikzcd}$$

So $f\alpha\pi = f$, and hence $\alpha\pi$ is an isomorphism since f is a cover. Therefore π is monic, and so $g = 0$.

$(2) \Rightarrow (1)$ Let M be any right R-module. Consider the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha"] & E^0 \arrow[rr, "\varphi"] \arrow[dr, "\beta"] & & E^1 \arrow[r, "\psi"] & N \arrow[r] & 0 \\
& & & C \arrow[ur, "i"] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where $\alpha : M \to E^0$ and $i : C \to E^1$ are special $(n, 2)$-injective preenvelopes respectively. Then N is $(n, 2)$-projective. Let $\theta : H \to N$ be an $(n, 0)$-injective cover with the unique mapping property. Then there exists $\delta : E^1 \to H$ such that $\psi = \theta\delta$. Thus $\theta\delta\varphi = \psi\varphi = 0 = \theta 0$, and hence $\delta\varphi = 0$, which implies that $\ker(\psi) = \mathrm{im}(\varphi) \subseteq \ker(\delta)$. Therefore there exists $\gamma : N \to H$ such that $\gamma\psi = \delta$, and so we get the following exact commutative diagram:

$$\begin{tikzcd}
& & & & H \arrow[d, shift right, "\theta"'] & \\
0 \arrow[r] & M \arrow[r] & E^0 \arrow[r, "\varphi"] & E^1 \arrow[ur, "\delta"] \arrow[r, "\psi"] & N \arrow[u, shift right, "\gamma"'] \arrow[r] & 0
\end{tikzcd}$$

Thus $\theta\gamma\psi = \psi$, and so $\theta\gamma = 1_N$ since ψ is epic. It follows that N is isomorphic to a direct summand of H, and hence N is $(n, 0)$-injective. So M is $(n, 2)$-injective. $\square$

It is well known that a right coherent ring R is right noetherian if and only if every (cyclic) right R-module is $(1, 0)$-projective if and only if every $(1, 0)$-injective right R-module is $(1, 0)$-projective if and only if every $(1, 0)$-injective right R-module is injective (see Mao and Ding, 2005a, Theorem 3.4 and Corollary 3.6).

We end this article by giving characterizations of those rings such that every right R-module is (n, d)-projective.

Proposition 4.10. *The following are equivalent for a right n-coherent ring R:*

(1) *Every right R-module is (n, d)-projective;*
(2) *Every finitely generated right R-module is (n, d)-projective;*
(3) *Every cyclic right R-module is (n, d)-projective;*
(4) *Every (n, d)-injective right R-module is (n, d)-projective;*
(5) *Every (n, d)-injective right R-module is injective.*

Moreover, if $d \geq 1$, then the above conditions are also equivalent to:

(6) *Every right R-module is $(n, 1)$-projective;*
(7) *Every (n, m)-injective right R-module is injective for any $m \geq 0$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3), and (1) $\Rightarrow$ (4) are obvious.

(3) $\Rightarrow$ (5) Let M be any (n, d)-injective right R-module and I any right ideal of R. Then $\mathrm{Ext}^1(R/I, M) = 0$ by (3). Thus M is injective, as desired.

(5) $\Rightarrow$ (1) follows from Theorem 3.9.

(4) $\Rightarrow$ (1) For any right R-module M, by Theorem 3.9, there is a short exact sequence $0 \to M \to N \to L \to 0$, where N is (n, d)-injective and L is (n, d)-projective. Thus M is (n, d)-projective by Lemma 3.4 (6) since N is (n, d)-projective by (4). Hence (1) follows.

(1) $\Rightarrow$ (6) holds by Lemma 3.4 (1).

(6) $\Rightarrow$ (7) By (6) and Theorem 3.9, every $(n, 1)$-injective right R-module is injective, and so every $(n, 0)$-injective right R-module is injective by Lemma 3.4 (1). If $m > 1$, then every (n, m)-injective right R-module is injective by Lemma 3.4 (5).

(7) $\Rightarrow$ (5) holds by letting $m = d$. □

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